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Strength of Materials

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PREFACE

An engineer always endeavours to design structural or machine members that are safe, durable and economical. To accomplish this, he has to evaluate the load-carrying capacity of the members so that they are able to withstand the various forces acting on them. The subject *Strength of Materials* deals with the strength, stability and rigidity of various structural or machine members such as beams, columns, shafts, springs, cylinders, etc. These days, a number of books on the subject are available in the market. However, it is observed that most of the books are feature-wise fine when considered on parameters like coverage of a topic, lucidity of writing, variety of solved and unsolved problems, quality of diagrams, etc., but usually, the students have to supplement a book with another book for one reason or the other. The present book aims to cover all good features in a single book.

The book is mainly aimed to be useful to degree-level students of mechanical and civil engineering as well as those preparing for AMIE and various other competitive examinations. However, diploma-level students will also find the book to be of immense use. The book will also benefit post-graduate students to some extent as it also contains some advance topics like bending of curved bars, rotating discs and cylinders, plastic bending and circular plates, etc. The salient features of the book are the following:

- A moderately concise and compact book covering all major topics
- Simple language to make it useful even to the average and weak students
- Logical and evolutionary approach in descriptions for better imagination and visualisation
- Physical concepts from simple and readily comprehensible principles
- Large number of solved examples
- Theoretical questions as well as sufficient number of unsolved problems at the end of each chapter
- Summary at the end of each chapter
- An appendix containing objective-type questions
- Another appendix containing important relations and results

It is expected that students using this book might have completed a course in applied mechanics. Chapters 1 and 2 introduce the concept of simple and compound stresses at a point. It is shown that an axial load may produce shear stresses along with normal stresses depending upon the section considered. The utility of Mohr's circle in transformation of stress at a point is also discussed. Chapter 3 explains the concept of strain energy that forms the basis of analysis in many cases. Chapters 4 to 8 are related to beams which may be simply supported, fixed at one or both ends or continuous having more than two supports. The analysis includes the computations of bending moment, shear force and bending and shear stresses under transverse loads. The concept of plastic deformations of beams beyond the elastic limit, being an advanced topic is taken up later and is discussed in Chapter 16. Sometimes, curved members such as rings and hooks are also loaded. Chapter 9 discusses the stresses developed in such members. The theory of torsion is developed in Chapter 10 which also includes its application to shafts transmitting power. The springs based on the same theory are discussed in the subsequent chapter. Columns are important members of structures. Chapter 12 discusses the equilibrium of columns and struts. However, the computation of stress in plane frame structures which is mostly included in the civil engineering curriculum is discussed later in Chapter 17. Some other important machine members include cylinders and spheres under internal or external pressures; flywheels, discs and cylinders which rotate while performing the required function; circular plates under concentrated or uniform loads. These topics are covered in chapters 13 to15. Chapter 18 discusses the properties of materials as well as the methods to determine the same.

Though students are expected to exert and solve the numerical problems given at the end of each chapter, hints to most of these are available at the publisher's website of the book for the benefit of average and weak students. However, full solutions of the unsolved problems are available to the faculty members at the same site. The facility can be availed by logging on to http://www.mhhe.com/rattan

In preparing the script, I relied heavily on the works of renowned authors whose writings are considered classics in the field. I am indeed indebted to them. I sincerely acknowledge the help of my many colleagues, who helped me in one form or the other in preparing this treatise. I also acknowledge the efforts of the editorial and production staff at Tata McGraw-Hill for taking pains in bringing out this book in an excellent format.

I am immensely thankful to the following reviewers who went through the manuscript and enriched it with their feedback.

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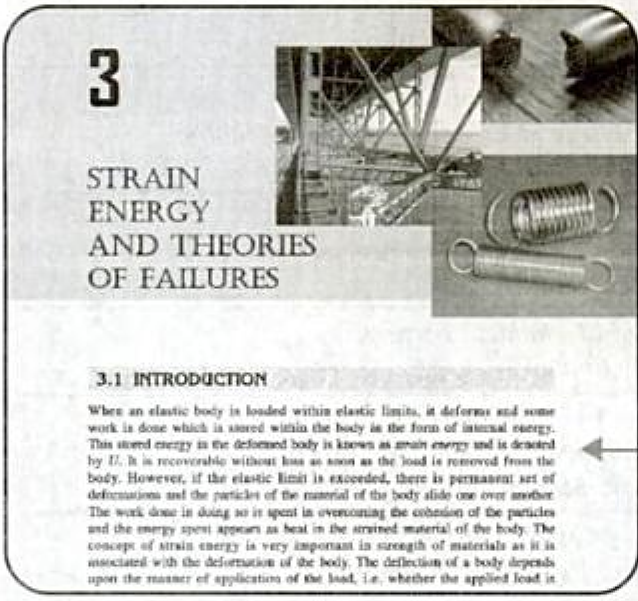
Finally, I am also indebted to my wife, Neena, and my children Ravneet and Jasmeet, for being patient with me while I went about the arduous task of preparing the manuscript. But for their sacrifice, I would not have been able to complete it in the most satisfying way.

A creation by a human being can never be perfect. A number of mistakes might have crept in the text. I shall be highly grateful to the readers and the users of the book for their uninhibited comments and pointing out the errors. Do feel free to contact me at ss_rattan@hotmail.com

S S Rattan

VISUAL WALKTHROUGH



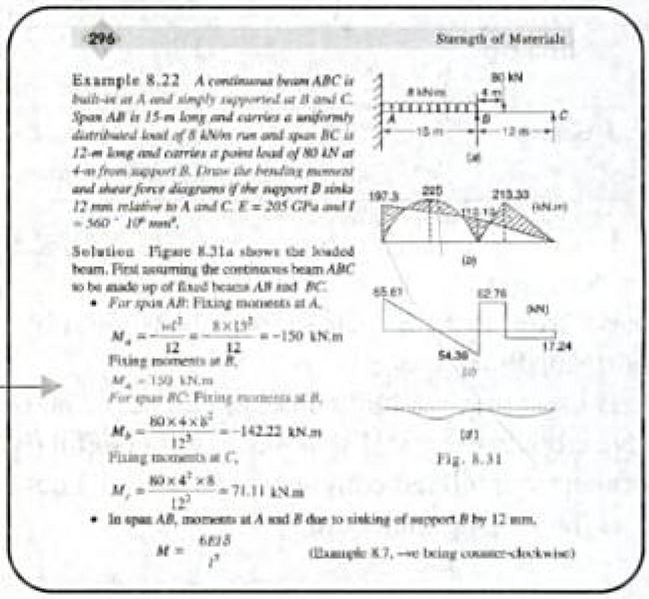
3

STRAIN ENERGY AND THEORIES OF FAILURES

3.1 INTRODUCTION

When an elastic body is loaded within elastic limits, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as *strain energy* and is denoted by *U*. It is recoverable without loss as soon as the load is removed from the body. However, if the elastic limit is exceeded, there is permanent set of deformations and the particles of the material of the body slide one over another. The work done in doing so is spent in overcoming the cohesion of the particles and the energy spent appears as heat in the strained material of the body. The concept of strain energy is very important in strength of materials as it is associated with the deformation of the body. The deflection of a body depends upon the manner of application of the load, i.e. whether the applied load is

Introduction at the beginning of each chapter sums up the aim and contents of the chapter.



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Example 8.22 *A continuous beam ABC is built-in at A and simply supported at B and C. Span AB is 15-m long and carries a uniformly distributed load of 8 kN/m run and span BC is 12-m long and carries a point load of 80 kN at 4-m from support B. Draw the bending moment and shear force diagrams if the support B sinks 12 mm relative to A and C. E = 205 GPa and I = 560 × 10⁶ mm⁴.*

Solution Figure 8.31a shows the loaded beam. First assuming the continuous beam ABC to be made up of fixed beams AB and BC.

- *For span AB*: Fixing moments at A,

$$M_a = -\frac{wl^2}{12} = -\frac{8 \times 15^2}{12} = -150 \text{ kN.m}$$

Fixing moments at B,

$M_a = 150$ kN.m

For span BC: Fixing moments at B,

$$M_b = -\frac{80 \times 4 \times 8^2}{12^2} = -142.22 \text{ kN.m}$$

Fixing moments at C,

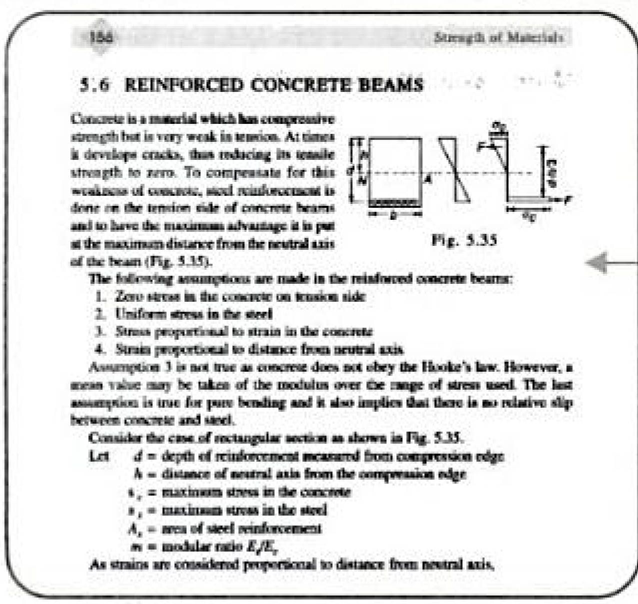
$$M_c = \frac{80 \times 4^2 \times 8}{12^2} = 71.11 \text{ kN.m}$$

- In span AB, moments at A and B due to sinking of support B by 12 mm,

$$M = \frac{6EI\delta}{l^2}$$ (Example 8.7, –ve being counter-clockwise)

Fig. 8.31

A variety of solved examples to reinforce the concepts.



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5.6 REINFORCED CONCRETE BEAMS

Concrete is a material which has compressive strength but is very weak in tension. At times it develops cracks, thus reducing its tensile strength to zero. To compensate for this weakness of concrete, steel reinforcement is done on the tension side of concrete beams and to have the maximum advantage it is put at the maximum distance from the neutral axis of the beam (Fig. 5.35).

Fig. 5.35

The following assumptions are made in the reinforced concrete beams:

1. Zero stress in the concrete on tension side
2. Uniform stress in the steel
3. Stress proportional to strain in the concrete
4. Strain proportional to distance from neutral axis

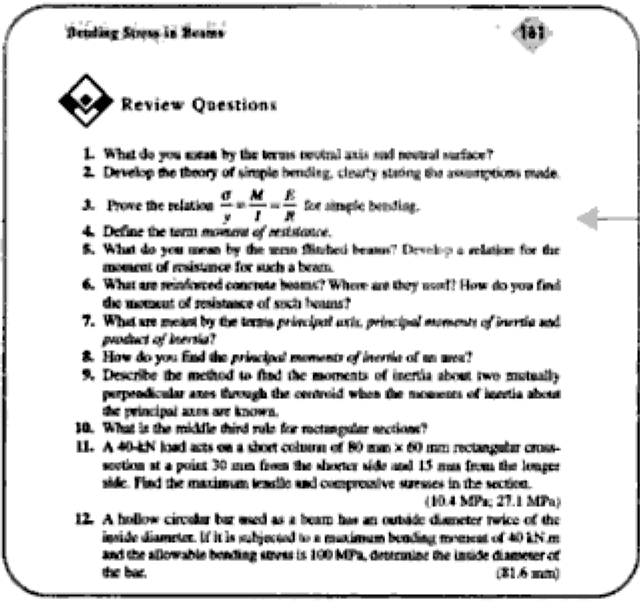
Assumption 3 is not true as concrete does not obey the Hooke's law. However, a mean value may be taken of the modulus over the range of stress used. The last assumption is true for pure bending and it also implies that there is no relative slip between concrete and steel.

Consider the case of rectangular section as shown in Fig. 5.35.

Let d = depth of reinforcement measured from compression edge
h = distance of neutral axis from the compression edge
s_c = maximum stress in the concrete
s_s = maximum stress in the steel
A_s = area of steel reinforcement
m = modular ratio E_s/E_c

As strains are considered proportional to distance from neutral axis,

Concise and comprehensive treatment of topics with emphasis on fundamental concepts.

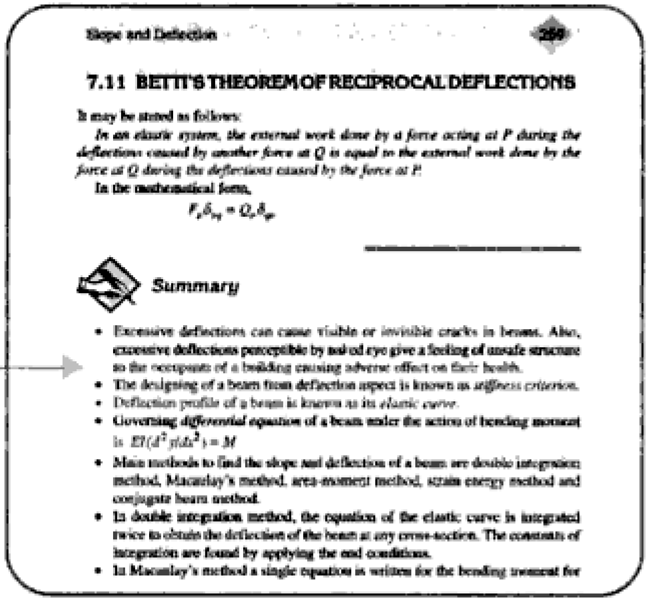


Bending Stress in Beams 181

Review Questions

1. What do you mean by the terms neutral axis and neutral surface?
2. Develop the theory of simple bending, clearly stating the assumptions made.
3. Prove the relation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$ for simple bending.
4. Define the term *moment of resistance*.
5. What do you mean by the term flitched beams? Develop a relation for the moment of resistance for such a beam.
6. What are reinforced concrete beams? Where are they used? How do you find the moment of resistance of such beams?
7. What are meant by the terms *principal axis, principal moments of inertia* and *product of inertia*?
8. How do you find the *principal moments of inertia* of an area?
9. Describe the method to find the moments of inertia about two mutually perpendicular axes through the centroid when the moments of inertia about the principal axes are known.
10. What is the middle third rule for rectangular sections?
11. A 40-kN load acts on a short column of 80 mm × 60 mm rectangular cross-section at a point 30 mm from the shorter side and 15 mm from the longer side. Find the maximum tensile and compressive stresses in the section. (10.4 MPa; 27.1 MPa)
12. A hollow circular bar used as a beam has an outside diameter twice of the inside diameter. If it is subjected to a maximum bending moment of 40 kN.m and the allowable bending stress is 100 MPa, determine the inside diameter of the bar. (81.6 mm)

A number of theoretical questions and unsolved problems for practice to widen the horizon of comprehension of the topic.



Slope and Deflection 259

7.11 BETTI'S THEOREM OF RECIPROCAL DEFLECTIONS

It may be stated as follows:

In an elastic system, the external work done by a force acting at P during the deflections caused by another force at Q is equal to the external work done by the force at Q during the deflections caused by the force at P.

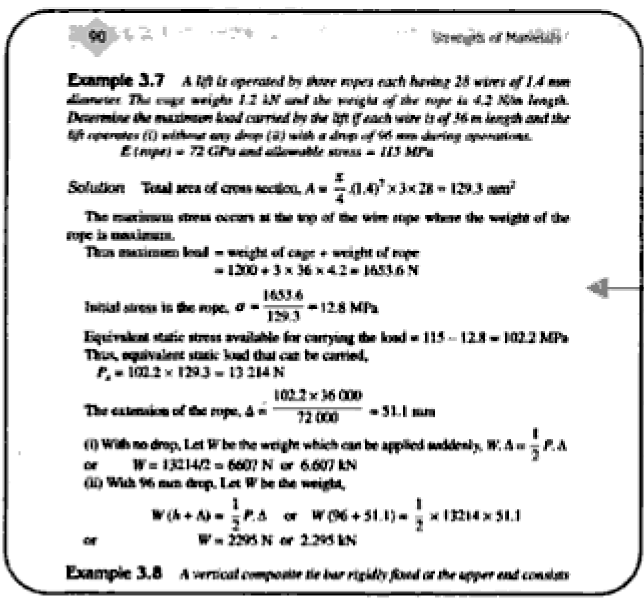
In the mathematical form,

$$P_p \delta_{pq} = Q_q \delta_{qp}$$

Summary

- Excessive deflections can cause visible or invisible cracks in beams. Also, excessive deflections perceptible by naked eye give a feeling of unsafe structure to the occupants of a building causing adverse effect on their health.
- The designing of a beam from deflection aspect is known as *stiffness criterion*.
- Deflection profile of a beam is known as its *elastic curve*.
- Governing *differential equation* of a beam under the action of bending moment is $EI(d^2y/dx^2) = M$
- Main methods to find the slope and deflection of a beam are double integration method, Macaulay's method, area-moment method, strain energy method and conjugate beam method.
- In double integration method, the equation of the elastic curve is integrated twice to obtain the deflection of the beam at any cross-section. The constants of integration are found by applying the end conditions.
- In Macaulay's method a single equation is written for the bending moment for

Summary at the end of each chapter recapitulates the inferences for quick revision.



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Example 3.7 *A lift is operated by three ropes each having 28 wires of 1.4 mm diameter. The cage weighs 1.2 kN and the weight of the rope is 4.2 N/m length. Determine the maximum load carried by the lift if each wire is of 36 m length and the lift operates (i) without any drop (ii) with a drop of 96 mm during operations.*

E (rope) = 72 GPa and allowable stress = 115 MPa

Solution Total area of cross section, $A = \frac{\pi}{4}(1.4)^2 \times 3 \times 28 = 129.3$ mm²

The maximum stress occurs at the top of the wire rope where the weight of the rope is maximum.

Thus maximum load = weight of cage + weight of rope
= 1200 + 3 × 36 × 4.2 = 1653.6 N

Initial stress in the rope, $\sigma = \frac{1653.6}{129.3} = 12.8$ MPa

Equivalent static stress available for carrying the load = 115 − 12.8 = 102.2 MPa

Thus, equivalent static load that can be carried,

$P_e = 102.2 \times 129.3 = 13\,214$ N

The extension of the rope, $\Delta = \frac{102.2 \times 36\,000}{72\,000} = 51.1$ mm

(i) With no drop, Let W be the weight which can be applied suddenly, $W.\Delta = \frac{1}{2}P.\Delta$

or $W = 13214/2 = 6607$ N or 6.607 kN

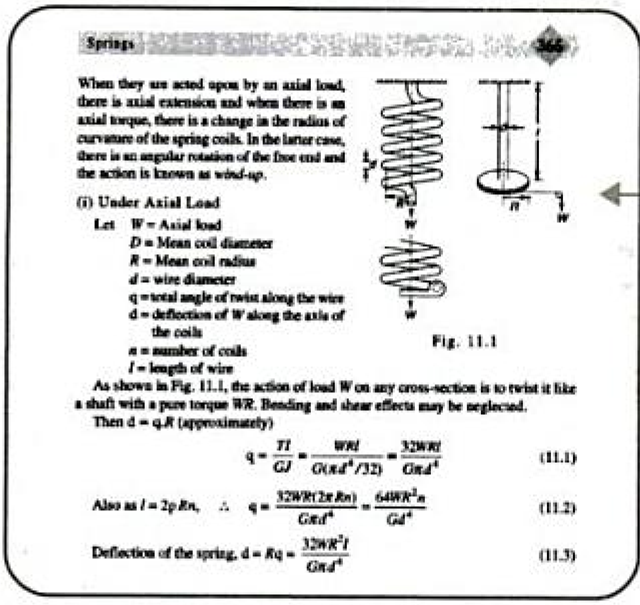
(ii) With 96 mm drop, Let W be the weight,

$$W(h + \Delta) = \frac{1}{2}P.\Delta \quad \text{or} \quad W(96 + 51.1) = \frac{1}{2} \times 13214 \times 51.1$$

or $W = 2295$ N or 2.295 kN

Example 3.8 *A vertical composite tie bar rigidly fixed at the upper end consists*

International system of units (SI) throughout the book for universal approval.



Springs 365

When they are acted upon by an axial load, there is axial extension and when there is an axial torque, there is a change in the radius of curvature of the spring coils. In the latter case, there is an angular rotation of the free end and the action is known as *wind-up*.

(i) Under Axial Load

Let W = Axial load
D = Mean coil diameter
R = Mean coil radius
d = wire diameter
q = total angle of twist along the wire
d = deflection of W along the axis of the coils
n = number of coils
l = length of wire

Fig. 11.1

As shown in Fig. 11.1, the action of load W on any cross-section is to twist it like a shaft with a pure torque WR. Bending and shear effects may be neglected.

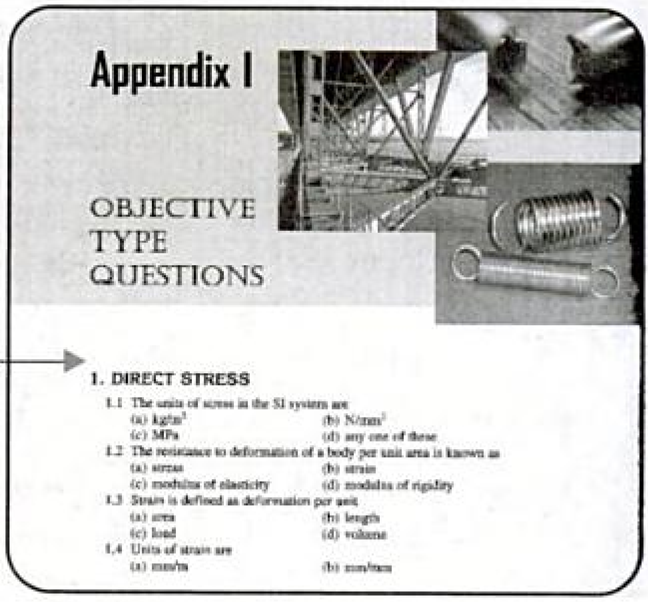
Then d = q.R (approximately)

$$q = \frac{Tl}{GJ} = \frac{WRl}{G(\pi d^4/32)} = \frac{32WRl}{G\pi d^4} \quad (11.1)$$

Also as l = 2pRn, $\therefore$ $$q = \frac{32WR(2\pi Rn)}{G\pi d^4} = \frac{64WR^2n}{Gd^4} \quad (11.2)$$

Deflection of the spring, d = Rq = $$\frac{32WR^2l}{G\pi d^4} \quad (11.3)$$

Simple diagrams for easy visualization of the explanations.



Appendix I

OBJECTIVE TYPE QUESTIONS

1. DIRECT STRESS

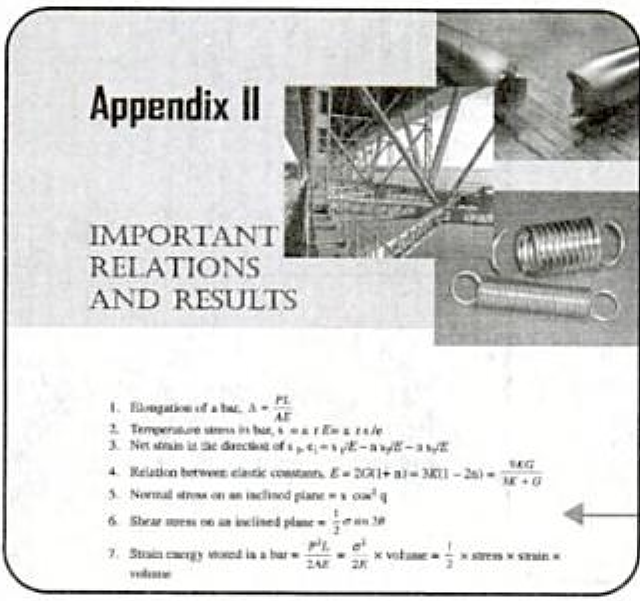
1.1 The units of stress in the SI system are
(a) kg/m² (b) N/mm²
(c) MPa (d) any one of these

1.2 The resistance to deformation of a body per unit area is known as
(a) stress (b) strain
(c) modulus of elasticity (d) modulus of rigidity

1.3 Strain is defined as deformation per unit
(a) area (b) length
(c) load (d) volume

1.4 Units of strain are
(a) mm/m (b) mm/mm

Appendix containing multiple choice questions to prepare for competitive examinations.



Appendix II

IMPORTANT RELATIONS AND RESULTS

1. Elongation of a bar, $\Delta = \frac{PL}{AE}$
2. Temperature stress in bar, s = a t E = a t s /e
3. Net strain in the direction of s₁, e₁ = s₁/E − n s₂/E − n s₃/E
4. Relation between elastic constants, $E = 2G(1+n) = 3K(1-2n) = \frac{9KG}{3K+G}$
5. Normal stress on an inclined plane = s cos² q
6. Shear stress on an inclined plane = $\frac{1}{2}\sigma \sin 2\theta$
7. Strain energy stored in a bar = $\frac{P^2L}{2AE} = \frac{\sigma^2}{2E}$ × volume = $\frac{1}{2}$ × stress × strain × volume

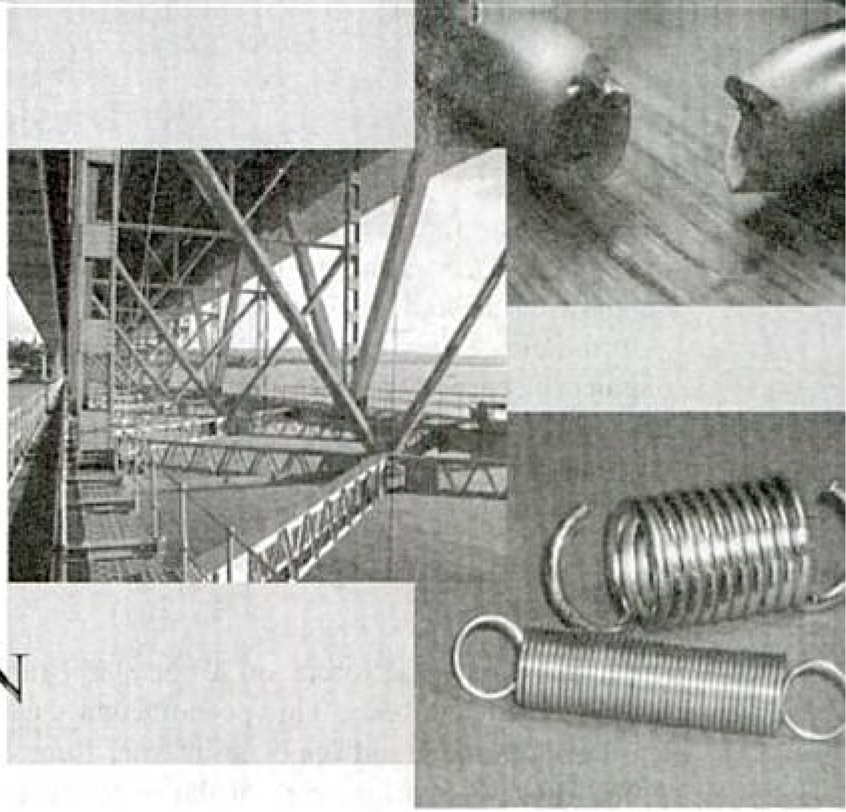
Appendix containing important relations for ready reference.

LIST OF SYMBOLS

a	area
A	area
b	width
B	width
d	diameter, height, depth
D	diameter
e	eccentricity
E	modulus of elasticity, Young's modulus
F	force
g	acceleration due to gravity
G	shear modulus, modulus of rigidity
h	height, distance
H	height
I	moment of inertia
l	length
j	number of joints
J	polar moment of inertia
k	torsional stiffness, stiffness of spring
K	Bulk modulus
l	length
L	length, load factor
m	mass, modular ratio, number of members
M	moment, bending moment, mass
n	number of coils
p	pressure, compressive stress
P	force, load
q	shear flow
r	radius
R	radius, reaction
s	length
S	shape factor
t	thickness, time, temperature
T	torque
u	strain energy density
U	strain energy, resilience
V	volume
w	rate of loading
W	force, weight, load
x, y, z	rectangular coordinates, distances
Z	section modulus
σ	direct stress
θ	angle
φ	angle, shear strain
α	angle, coefficient of thermal expansion
δ	increment of quantity, deflection, extension
τ	shear stress
Δ	elongation
ε	direct strain
π	3.1416
ν	Poisson's ratio
ψ	angle
γ	angle
ω	angular velocity
ρ	density

1

SIMPLE STRESS AND STRAIN



1.1 INTRODUCTION

External forces acting on individual structural or machine members of an engineering design are common. An engineer always endeavours to have such a design so that these are safe, durable and economical. Thus load carrying capacity of the members being designed is of paramount importance to know their dimensions to have the minimum cost. The subject *Strength of Materials* deals with the strength or the load-carrying capacity of various members such as beams and columns. It also considers their stability and rigidity. *Theory of structures* involves the application of these principles to structures made up of beams, columns, slabs and arches.

Force acting on a body is termed as *load*. A *concentrated load* is also known as a *point load* and a distributed load over a length is known as *distributed load*. Distributed load of constant value is called *uniformly distributed load*. If a structure as a whole is in equilibrium, its members are also in equilibrium individually which implies that the resultant of all the forces acting on a member must be zero. However, the forces acting on a body tend to deform or torn the body. For example, a load P acting on a body tends to pull it apart (Fig. 1.1a). This type of pull may also be applied if one end of the body is fixed (Fig. 1.1b). In this case, the balancing force is provided by the reaction of the fixed end. Such type of pulling force is known as *tension* or *tensile force*. A tensile force tends to increase the length and decrease the cross-section of the body.

In a similar way, a force tending to push or compress a body is known as *compression* or *compressive force* which tends to shorten the length (Fig. 1.1c).

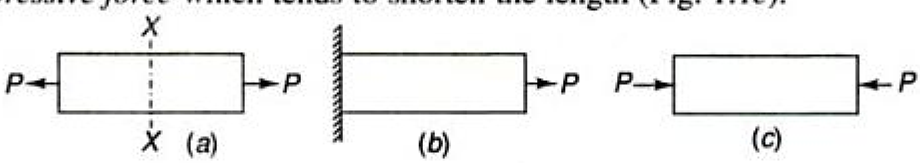


Fig. 1.1

Usually, the forces acting on a body along the longitudinal axis are known as *direct* or *axial forces* and the forces acting normal to the longitudinal axis of a body are known as *transverse* or *normal* forces.

In the elementary theory of analysis, a material subjected to external forces is assumed to be perfectly elastic, i.e. the deformations caused to the body totally disappear as soon as the load or forces are removed. Other assumptions are that the materials are *isotropic* (same properties in all directions) and *homogeneous* (same properties anywhere in the body).

1.2 STRESS

The applied external forces on a body are transmitted to the supports through the material of the body. This phenomenon tends to deform the body and causes it to develop equal and opposite internal forces. These *internal forces* by virtue of cohesion between particles of the material tend to resist the deformation. The magnitude of the internal resisting forces is equal to the applied forces but the direction is opposite.

Let the member shown in Fig. 1.1*a* be cut through the section *X-X* as shown in Fig. 1.2. Now, each segment of the member is in equilibrium under the action of force *P* and the internal resisting force. The resisting force per unit area of the surface is known as *intensity of stress* or simply *stress* and is denoted by σ. Thus if the load *P* is assumed as uniformly distributed over a sectional area *A*, then the stress σ is given by

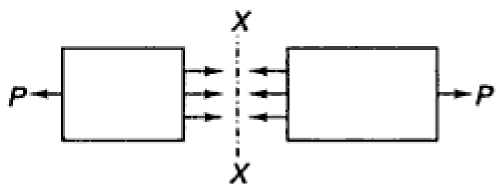


Fig. 1.2

$$\sigma = P/A \qquad (1.1)$$

However, if the intensity of stress is not uniform throughout the body, then the stress at any point is defined as

$$\sigma = \delta P/\delta A$$

where δA = Infinitesimal area of cross-section

and δP = Load applied on area δA

The stress may be tensile or compressive depending upon the nature of forces applied on the body.

Stress at the elastic limit is usually referred as *proof stress.*

Units

The unit of stress is N/m^2 or Pascal (Pa). However, this is a very small unit, almost the stress due to placing an apple on an area of 1 m^2. Thus it is preferred to express stress in units of MN/m^2 or MPa.

$$1\ MN/m^2 = 1\ MPa = 1 \times 10^6\ N/m^2 = 1\ N/mm^2$$

Also $$1\ GPa = 1000\ MPa = 1000\ N/mm^2 = 1\ kN/mm^2$$

In numerical problems, it is always convenient to express the units of stress mentioned in MPa and GPa in the form of N/mm^2.

1.3 SHEAR STRESS

When two equal and opposite parallel forces not in the same line act on two parts of a body, then one part tends to slide over or shear from the other across any section and the stress developed is termed as *shear stress.* In Fig. 1.3*a* and *b*, the material is sheared along any section *X-X* whereas in a riveted joint (Fig.1.3*c*), the shearing is across the rivet diameter.

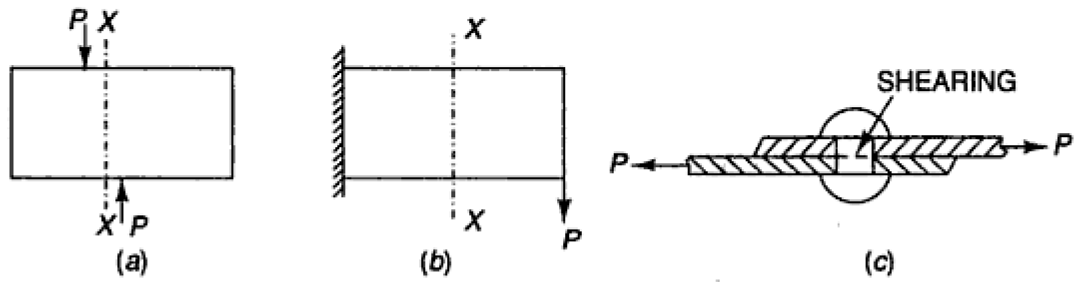


Fig. 1.3

If P is the force applied and A is the area being sheared, then the intensity of shear stress is given by

$$\tau = P/A \tag{1.2}$$

and if the intensity of shear stress varies over an area,

$$\tau = \delta P/\delta A$$

Remember that shear stress is always tangential to the area over which it acts.

Complimentary Shear Stress

Consider an infinitely small rectangular element *ABCD* under shear stress of intensity τ acting on planes *AD* and *BC* as shown in Fig. 1.4*a*. It is clear from the figure that the shear stress acting on the element will tend to rotate the block in the clockwise direction.

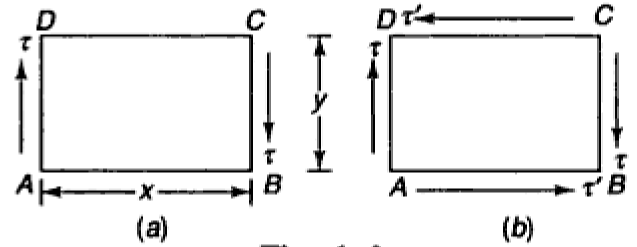


Fig. 1.4

As there is no other force acting on the element, static equilibrium of the element can only be attained if another couple of the same magnitude is applied in the counter-clockwise direction. This can be achieved by having shear stress of intensity τ' on the faces *AB* and *CD* (Fig. 1.4*b*).

Assuming x and y to be the lengths of the sides *AB* and *BC* of the rectangular element and a unit thickness perpendicular to the figure,

The force of the given couple = $\tau.(y.1)$

The moment of the given couple = $(\tau.y).x$
Similarly,
The force of balancing couple = $\tau'.(x.1)$
The moment of balancing couple = $(\tau'.x).y$
For equilibrium, equating the two,

$$(\tau.y).x = (\tau'.x).y \quad \text{or} \quad \tau = \tau'$$

which shows that the magnitude of the balancing shear stresses is the same as of the applied stresses. The shear stresses on the transverse pair of faces are known as *complimentary shear stresses*. Thus every shear stress is always accompanied by an equal complimentary shear stress on perpendicular planes.

Owing to the characteristic of complimentary shear stresses for the equilibrium of members subjected to shear stresses, near a free boundary on which no external force acts, the shear stress must follow a direction parallel to the boundary. This is because any component of the shear force perpendicular to the surface will find no complimentary shear stress on the boundary plane. The presence of complimentary shear stress may cause an early failure of *anisotropic materials* such as timber which is weaker in shear along the grain than normal to the grain.

1.4 STRAIN

The deformation of a body under a load is proportional to its length. To study the behaviour of a material, it is convenient to study the deformation per unit length of a body than its total deformation. The elongation per unit length of a body is known as *strain* and is denoted by Greek symbol ε. If Δ is the elongation of a body of length L, the strain ε is given by

$$\varepsilon = \Delta/L \tag{1.3}$$

As it is a ratio of similar quantities, it is dimensionless.

Shear Strain

A rectangular element of a body is distorted by shear stresses as shown in Fig. 1.5. If the lower surface is assumed to be fixed, the upper surface slides relative to the lower surface and the corner angles are altered by angle φ. *Shear strain* is defined as the change in the right angle of the element measured in radians and is dimensionless.

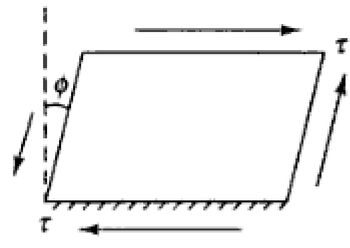


Fig. 1.5

1.5 MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

For elastic bodies, the ratio of stress to strain is constant and is known as *Young's Modulus* or the *Modulus of Elasticity* and is denoted by E, i.e.,

$$E = \sigma/\varepsilon \tag{1.4}$$

Strain has no units as it is a ratio. Thus E has the same units as stress.

The materials that maintain this ratio are said to obey *Hook's law* which states that within elastic limits, strain is proportional to the stress producing it. The elastic limit of a material is determined by plotting a tensile test diagram (Refer section 1.15).

Young's modulus is the stress required to cause a unit strain. As a unit strain means elongation of a body equal to original length (since $\varepsilon = \Delta/L$), this implies that Young's modulus is the stress or the force required per unit area to elongate the body by its original size or to causes the length to be doubled. However, for most of the engineering materials, the strain does not exceed 1/1000. Obviously, mild steel has a much higher value of Young's modulus E as compared to rubber.

Similarly, for elastic materials, the shear strain is found to be proportional to the applied shear stress within the elastic limit. *Modulus of rigidity* or *shear modulus* denoted by G is the ratio of shear stress to shear strain, i.e.

$$G = \tau/\varphi \qquad (1.5)$$

1.6 ELONGATION OF A BAR

An expression for the elongation of a bar of length L and cross-sectional area A under the action of a force P is obtained below:

As $\quad E = \dfrac{\sigma}{\varepsilon} \qquad \therefore \qquad \varepsilon = \dfrac{\sigma}{E} \quad \text{or} \quad \dfrac{\Delta}{L} = \dfrac{P}{AE}$

Thus elongation of a bar of length L, $\quad \Delta = \dfrac{PL}{AE} \qquad (1.6)$

1.7 PRINCIPLE OF SUPERPOSITION

The principle of superposition states that if a body is acted upon by a number of loads on various segments of a body, then the net effect on the body is the sum of the effects caused by each of the loads acting independently on the respective segment of the body. Thus each segment can be considered for its equilibrium. This is done making a diagram of the segment alongwith various forces acting on it. This diagram is generally referred as *free body diagram*. The principle of superposition is applicable to all parameters like stress, strain and deflection. However, it is not applicable to materials with non-linear stress-strain characteristics which do not follow Hook's law.

Example 1.1 *A steel bar of 25-mm diameter is acted upon by forces as shown in Fig. 1.6a. What is the total elongation of the bar? Take E = 190 GPa.*

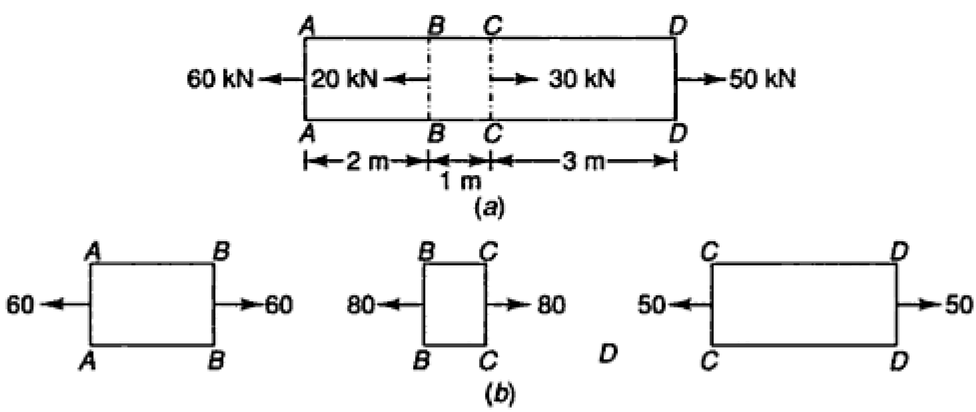


Fig. 1.6

Solution Area of the section = $\frac{\pi}{4}(25)^2 = 490.88\ \text{mm}^2$, $E = 190$ GPa
$= 190\,000\ \text{N/mm}^2$

Forces in various segments are considered by taking free-body diagram of each segment as follows (Fig. 1.6*b*):

Segment AB: At section *AA*, it is 60 kN tensile and for force equilibrium of this segment, it is to be 60 kN tensile at *BB* also.
Segment BC:
Force at section *BB* = 60 kN (as above) + 20 kN (tensile force at section *BB*)
= 80 kN (tensile) = Force at section *CC*

Segment CD:
Force at section *CC* = 80 kN (as above) – 30 kN (compressive force at section *CC*)
= 50 kN (tensile) = Force at section *DD*

Elongation is given by, $\Delta = \frac{PL}{AE}$

$$= \frac{1}{490.88 \times 190\,000}(60\,000 \times 2000 + 80\,000 \times 1000 + 50\,000 \times 3000) = 3.75 \text{ mm}$$

Example 1.2 *A steel circular bar has three segments as shown in Fig. 1.7a. Determine*

(i) the total elongation of the bar
(ii) the length of the middle segment to have zero elongation of the bar
(iii) the diameter of the last segment to have zero elongation of the bar
Take E = 205 GPa.

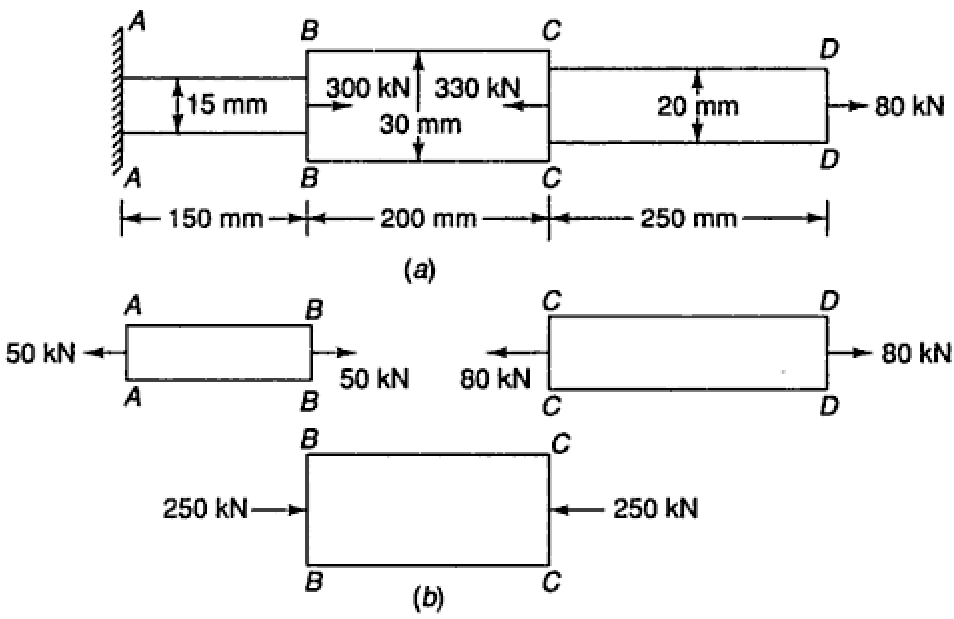


Fig. 1.7

Solution Forces in various segments (Fig. 1.7*b*):

(i) *Segment CD*: At section *DD*, it is 80 kN tensiles and for force equilibrium of this segment, at *CC* also it is to be 80 kN tensile.

Segment BC:

Force at section *CC* = 80 kN (as above) – 330 kN (compressive force at section *CC*)
= – 250 kN (compressive) = Force at section *BB*

Segment AB:

Force at section *BB* = – 250 kN (as above) + 300 kN (tensile force at section *BB*)
= 50 kN (tensile) = Force at section *AA*

Total elongation,

$$\Delta = \frac{1}{(\pi/4) \times 205\,000}\left(\frac{50\,000 \times 150}{15^2} - \frac{250\,000 \times 200}{30^2} + \frac{80\,000 \times 250}{20^2}\right)$$

$$= \frac{1}{161\,007}(33\,333.3 - 55\,555.5 + 50\,000) = 0.173 \text{ mm}$$

(ii) Let the length of the middle segment be *L* to have zero elongation of the bar.

Then $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - \frac{250\,000 \times L}{30^2} + 50\,000\right) = 0$$

or $$L = \frac{30^2}{250\,000} \times 83\,333.3 = 300 \text{ mm}$$

(iii) Let the diameter of the last segment be *d* to have zero elongation of the bar.

∴ $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - 55\,555.5 + \frac{80\,000 \times 250}{d^2}\right) = 0$$

$$d^2 = \frac{80\,000 \times 250}{22\,222.2} = 900 \quad \text{or} \quad d = 30 \text{ mm}$$

1.8 BARS OF TAPERING SECTION

Bars of tapering section can be of conical section or of trapezoidal section with uniform thickness.

Conical Section

Consider a bar of conical section under the action of axial force *P* as shown in Fig. 1.8.

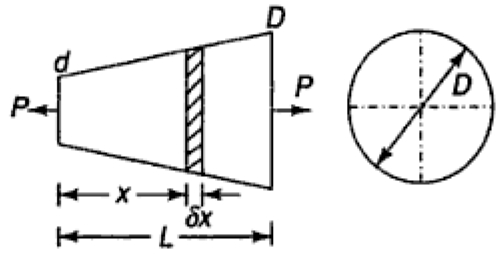


Fig. 1.8

Let D = diameter at the larger end
d = diameter at the smaller end
L = length of the bar
E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end.

The diameter at a distance x from the small end $= d + \dfrac{D - d}{L}.x$

The extension of a small length

$$= \frac{P.\delta x}{\frac{\pi}{4}\left(d + \frac{D-d}{L}x\right)^2 .E} \qquad \text{.....} \left(\Delta = \frac{PL}{AE}\right)$$

Extension of the whole rod $= \int_0^L \frac{4P}{\pi\left(d + (D-d)x/L\right)^2 .E}.dx$

$$= \frac{4P}{\pi E}\int_0^L \left(d + \frac{D-d}{L}.x\right)^{-2}.dx = -\frac{4P}{\pi E}.\frac{L}{(D-d)}\left(\frac{1}{\left(d + (D-d)x/L\right)}\right)_0^L$$

$$= \frac{4PL}{\pi E(D-d)}\left(\frac{1}{d} - \frac{1}{D}\right) = \frac{4PL}{\pi E(D-d)}\left(\frac{D-d}{dD}\right) = \frac{4PL}{\pi EdD} \qquad (1.7)$$

Trapezoidal Section of Uniform Thickness

Let

B = width at the larger end b = width at the smaller end

t = thickness of the section L = length of the bar

E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end of the rod (Fig.1.9).

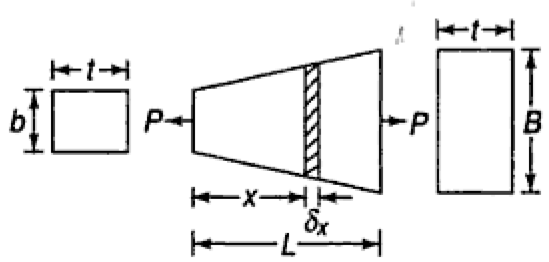


Fig. 1.9

The width at a distance x from the small end

$$= b + \frac{B-b}{L}.x = b + kx$$

....[Taking $k = (B-b)/L$]

The area of cross-section at this distance $= (b + kx).t$

The extension of the small length $= \frac{P.\delta x}{(b + kx)t.E}$

Extension of the whole rod

$$= \int_0^L \frac{P}{(b+kx)t.E}.dx = \frac{P}{tE}\int_0^L \frac{1}{(b+kx)}.dx$$

$$= \frac{P}{tE}\frac{1}{k}\left[\log_e(b+kx)\right]_0^L = \frac{P}{ktE}\left(\log_e \frac{b+kL}{b}\right) = \frac{P}{ktE}\log_e \frac{B}{b} \qquad (1.8)$$

$$\text{.......}\left(b + kL = b + \frac{B-b}{L}.L = B\right)$$

1.9 ELONGATION DUE TO SELF-WEIGHT

The elongation due to self-weight of bars of rectangular and conical sections may be considered as follows:

Rectangular Section

Consider a bar hanging freely under its own weight as shown in Fig. 1.10.

Consider a small length δx of the bar at a distance x from the free end.

Let A = area of cross-section of the bar
w = weight per unit length of the bar
W = weight of the whole bar = wL
W_x = weight of the bar below the small section = wx

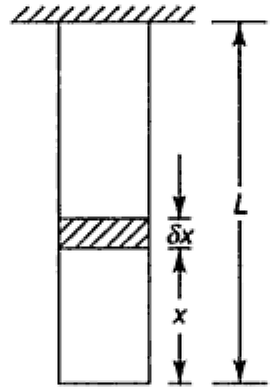


Fig. 1.10

The extension of a small length = $\dfrac{W_x.\delta x}{A.E} = \dfrac{wx.\delta x}{A.E}$

Extension of the whole rod

$$= \int_0^L \frac{wx}{AE}.dx = \frac{w}{AE}\left(\frac{x^2}{2}\right)_0^L = \frac{wL^2}{2AE} = \frac{wL.L}{2AE} = \frac{WL}{2AE} \qquad (1.9)$$

= deformation due to a weight W at the lower end/2

Thus the deformation of the bar under its own weight is equal to half the deformation due to a direct load equal to the weight of the body applied at the lower end.

Conical Section

Consider a small length δx of the bar at a distance x from the free end (Fig. 1.11).

Let A = area of cross-section at the small length
w = weight per unit volume of the bar
W_x = weight of the bar below the section = $wAx/3$

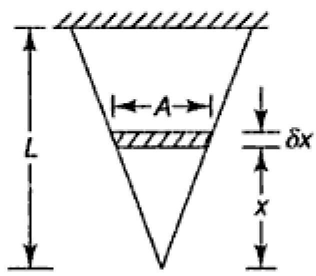


Fig. 1.11

The extension of a small length = $\dfrac{W_x.\delta x}{A.E} = \dfrac{wAx.\delta x}{3A.E}$

Extension of the whole rod = $\displaystyle\int_0^L \frac{wAx}{3AE}.dx = \frac{w}{3E}\int_0^L x.dx$

$$= \frac{wL^2}{6E} \qquad (1.10)$$

Comparing it with Eq. 1.9, this elongation is one-third that of the rectangular section of the same length under own weight of the bar.

1.10 COLUMN OF UNIFORM STRENGTH

Let a bar of varying cross-sectional area be acted upon by a load P as shown in Fig. 1.12. Consider a small length dx at a distance x from the top.

Let A = area at distance x

$A + dA$ = area at distance $x + dx$

w = weight per unit volume of the bar

Considering the balance of forces acting on the small length,

$\sigma (A + dA) = \sigma A$ + weight of the small length dx of the bar

or $\sigma (A + dA) = \sigma A + wAdx$

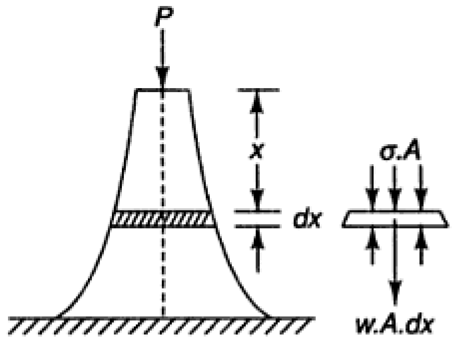


Fig. 1.12

or $\sigma . dA = wAdx$ or $\frac{dA}{A} = \frac{w}{\sigma} dx$

Integrating both sides, $\log_e A = \frac{w}{\sigma} x + C$

At the top, where $x = 0$, let Area $A = a$

Then, $\log_e a = 0 + C$ or $C = \log_e a$

Thus $\log_e A = \frac{w}{\sigma} + \log_e a$ or $\log_e \frac{A}{a} = \frac{w}{\sigma} x$

or $\frac{A}{a} \; e^{wx/\sigma}$

or $A = ae^{wx/\sigma}$ (1.11)

1.11 STATICALLY INDETERMINATE SYSTEMS

When a system comprises two or more members of different materials, the forces in various members cannot be determined by the principle of statics alone. Such systems are known as *statically indeterminate systems*. In such systems, additional equations are required to supplement the equations of statics to determine the unknown forces. Usually, these equations are obtained from deformation conditions of the system and are known as *compatibility equations*. A compound bar is a case of an indeterminate system and is discussed below:

Compound Bar

A bar consisting of two or more bars of different materials in parallel is known as a *composite* or *compound bar*. In such a bar, the sharing of load by each can be found by applying equilibrium and the compatibility equations.

Consider the case of a solid bar enclosed in a hollow tube as shown in Fig. 1.13. Let the subscripts 1 and 2 denote the solid bar and the hollow tube respectively.

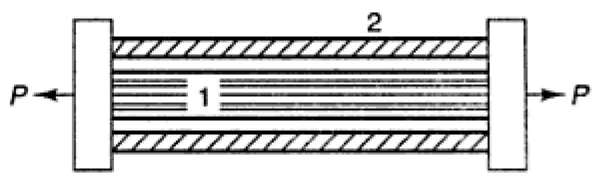


Fig. 1.13

Equilibrium equation As the total load must be equal to the load taken by individual members,

$$P = P_1 + P_2 \qquad \text{(i)}$$

Compatibility equation The deformation of the bar must be equal to the tube.

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} \quad \text{or} \quad P_1 = \frac{P_2 A_1 E_1}{A_2 E_2} \tag{ii}$$

Inserting (ii) in (i),

$$P = \frac{P_2 A_1 E_1}{A_2 E_2} + P_2 = \frac{P_2 A_1 E_1 + P_2 A_2 E_2}{A_2 E_2} = \frac{P_2 (A_1 E_1 + A_2 E_2)}{A_2 E_2}$$

or
$$P_2 = \frac{P.A_2 E_2}{A_1 E_1 + A_2 E_2} \tag{1.12}$$

Similarly,
$$P_1 = \frac{P.A_1 E_1}{A_1 E_1 + A_2 E_2} \tag{1.13}$$

Example 1.3 *Three equally spaced rods in the same vertical plane support a rigid bar AB. Two outer rods are of brass, each 600-mm long and of 25-mm diameter. The central steel rod is 800-mm long and 30 mm in diameter. Determine the forces in the bars due to an applied load of 120 kN through the mid-point of the bar. The bar remains horizontal after the application of load. Take $E_s/E_b = 2$.*

Solution Refer Fig. 1.14.

As the bar remains horizontal after the application of load, the elongation of each of the brass bars and of the steel bar are the same.

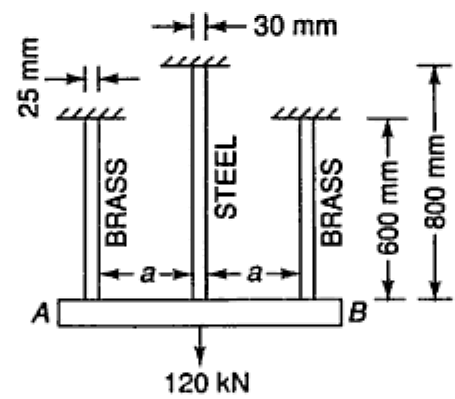


Fig. 1.14

From compatibility equation, $\Delta_b = \Delta_s$

or
$$\frac{P_b L_b}{A_b E_b} = \frac{P_s L_s}{A_s E_s}$$

or
$$P_b = \frac{L_s}{L_b} \cdot \frac{E_b}{E_s} \left(\frac{d_b}{d_s} \right)^2 P_s$$

$$= \frac{800}{600} \cdot \frac{1}{2} \left(\frac{25}{30} \right)^2 P_s$$

or $\quad P_b = 0.463\, P_s$

From equilibrium equation, $2P_b + P_s = P$

or $\; 2 \times 0.463\, P_s + P_s = 120 \quad$ or $\quad 1.926\, P_s = 120$

or $\quad P_s = 62.3$ kN and $P_b = 28.84$ kN

Example 1.4 *Three equidistant vertical rods each of 20-mm diameter support a load of 25 kN in the same plane as shown in Fig. 1.15. Initially, all the rods are adjusted to share the load equally. Neglecting any chance of buckling, and taking $E_s = 205$ GPa and $E_c = 100$ GPa, determine the final stresses when a further load of 20 kN is added.*

Solution $A = (\pi/4)\,20^2 = 100\,\pi\ \text{mm}^2$

Initially, the stress in each rod $= \dfrac{25\,000}{100\pi \times 3} = 26.53$ MPa

On adding a further load of 20 kN, let the increase of stress in the steel rod be σ_s and in the copper rod σ_c.

Then from equilibrium equation, the additional load P is

$$(2\sigma_s + \sigma_c)A = P \text{ or } (2\sigma_s + \sigma_c) \times 100\pi = 20\,000 \quad \text{(i)}$$

From compatibility equation, $\Delta_c = \Delta_s$

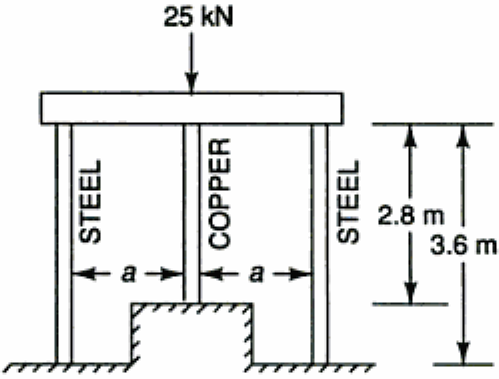


Fig. 1.15

$$\frac{\sigma_c L_c}{E_c} = \frac{\sigma_s L_s}{E_s}$$

$$\text{or} \quad \sigma_c = \frac{L_s}{L_c}\cdot\frac{E_c}{E_s}\sigma_s = \frac{3.6}{2.8} \times \frac{100\,000}{205\,000}\sigma_s \text{ or } \sigma_c = 0.627\sigma_s$$

Inserting this value of σ_c in (i)

$\therefore$ $(2\sigma_s + 0.627\,\sigma_s) \times 100\pi = 20\,000$

or $2.627\,\sigma_s = 63.662$

or $\sigma_s = 24.23$ MPa and $\sigma_c = 15.19$ MPa

Final stress in steel rod = 24.23 + 26.53 = 50.76 MPa

Final stress in copper rod = 15.19 + 26.53 = 41.72 MPa

Example 1.5 *A steel rod of 16-mm diameter passes through a copper tube of 20 mm internal diameter and of 32-mm external diameter. The steel rod is fitted with nuts and washers at each end. The nuts are tightened till a stress of 24 MPa is developed in the steel rod. A cut is then made in the copper tube for half the length to remove 2 mm from its thickness. Assuming the Young's modulus of steel to be twice that of copper, determine*

(i) the stress existing in the steel rod.

(ii) the stress in the steel rod if a compressive load of 4 kN is applied to the ends of the steel rod.

Solution Refer Fig. 1.16.

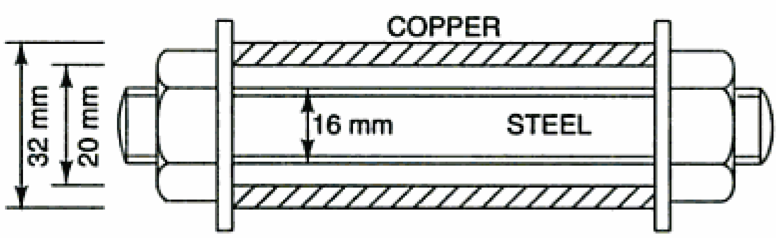


Fig. 1.16

$$A_s = \frac{\pi}{4} \times 16^2 = 64\pi \text{ mm}^2$$

and $$A_c = \frac{\pi}{4}(32^2 - 20^2) = 156\pi \text{ mm}$$

On tightening the nut, the steel rod is elongated and the stress induced is tensile whereas the tube is shortened and the stress is compressive.

Let σ_{s1} = stress in the steel rod = 24 MPa

σ_{c1} = stress in the copper tube

From equilibrium equation

Push on copper tube = Pull on steel rod

$$\sigma_{c1} \times A_c = \sigma_{s1} \times A_s \quad \text{or} \quad \sigma_{c1} \times 156\pi = 24 \times 64\pi$$

or σ_{c1} = 9.846 MPa (compressive)

(i) When the copper tube is reduced in diameter,

$A_{c'}$ = reduced area of cross-section of the tube = $\frac{\pi}{4}(28^2 - 20^2) = 96\pi$

Let σ_{s2} = stress in the steel rod

$\sigma_{c2'}$ = stress in the reduced section of tube

and σ_{c2} = stress in the remaining section of tube

From equilibrium equation,

Force in each section of copper tube as well as in the steel rod are to be equal

i.e. $$\sigma_{c2} \times 156\pi = \sigma_{c2'} \times 96\pi = \sigma_{s2} \times 64\pi \quad \text{(i)}$$

$$\sigma_{c2} = 0.4103\ \sigma_{s2} \text{ and } \sigma_{c2'} = 0.6667\ \sigma_{s2}$$

From compatibility equation,

When the cross-section of the tube is reduced, the change in length of the rod as well as of the tube is to of the same nature, i.e. either the length of both is increased or decreased. Let us assume that the length of each is reduced which means a reduction of tensile stress in the rod and increase of compressive stress in the tube.

Thus reduction in length of steel rod = reduction in length of copper tube

$$\frac{\sigma_{s1} - \sigma_{s2}}{E_s}.L = \frac{\sigma_{c2} - \sigma_{c1}}{E_c}.\frac{L}{2} + \frac{\sigma'_{c_2} - \sigma_{c1}}{E_c}.\frac{L}{2}$$

or $$\sigma_{s1} - \sigma_{s2} = \sigma_{c2} + \sigma'_{c2} - 2\sigma_{c1} \quad \ldots\ldots \quad (E_s = 2E_c) \quad \text{(ii)}$$

or $$24 - \sigma_{s2} = 0.4103\ \sigma_{s2} + 0.6667\sigma_{s2} - 2 \times 9.846$$

or $$2.077\sigma_{s2} = 43.692 \quad \text{or} \quad \sigma_{s2} = 21.036 \text{ MPa}$$

As the stress in the steel rod is decreased from 24 MPa to 21.036 MPa, the assumption of reduction of the length of the two is correct. In case, the lengths are assumed to be increased, the stress in the steel rod is increased and in the copper tube decreased. The equation formed would have been

$$\frac{\sigma_{s2} - \sigma_{s1}}{E_s}.L = \frac{\sigma_{c1} - \sigma_{c2}}{E_s}.\frac{L}{2} + \frac{\sigma_{c1} - \sigma'_{c2}}{E_s}.\frac{L}{2}$$

and the result would have been the same i.e. σ_{s2} = 21.036 MPa which would have indicated that the length actually would be reduced due to decrease in the stress of steel rod.

(ii) When a compressive load of 4 kN is applied to the ends of the steel rod, the length of the rod is further reduced.

Equilibrium equation

$$\sigma_{c3} \times 156\pi = \sigma'_{c3} \times 96\pi = \sigma_{s3} \times 64\pi + 6000 \quad \text{[as in (i)]}$$

or $$\sigma_{c3} \times 156 = \sigma'_{c3} \times 96 = \sigma_{s3} \times 64 + 1909.9$$

or $$\sigma_{c3} = 0.4103\ \sigma_{s3} + 12.243$$

and $$\sigma'_{c3} = 0.6667\ \sigma_{s3} + 19.895$$

Compatibility equation

$$\sigma_{s1} - \sigma_{s3} = \sigma_{c3} + \sigma'_{c3} - 2\sigma_{c1} \quad \text{[as in (ii)]}$$

or $$24 - \sigma_{s3} = 0.4103\sigma_{s3} + 12.243 + 0.6667\sigma_{s3} + 19.895 - 2 \times 9.846$$

or $$2.077\sigma_{s3} = 11.554 \quad \text{or} \quad \sigma_{s3} = 5.56 \text{ MPa}$$

Example 1.6 *A round steel rod supported in a recess is surrounded by a co-axial brass tube as shown in Fig. 1.17. The level of the upper end of the rod is 0.08 mm below that of the tube. Determine:*

(i) the magnitude and direction of the maximum permissible axial load which can be applied to a rigid plate resting on the top of the tube. The permissible values of the compressive stresses are 105 MPa for steel and 75 MPa for brass.

(ii) the amount by which the tube is shortened by a load if the compressive stresses in the steel and the brass are the same.

Take E_s = 210 GPa and E_b = 105 GPa.

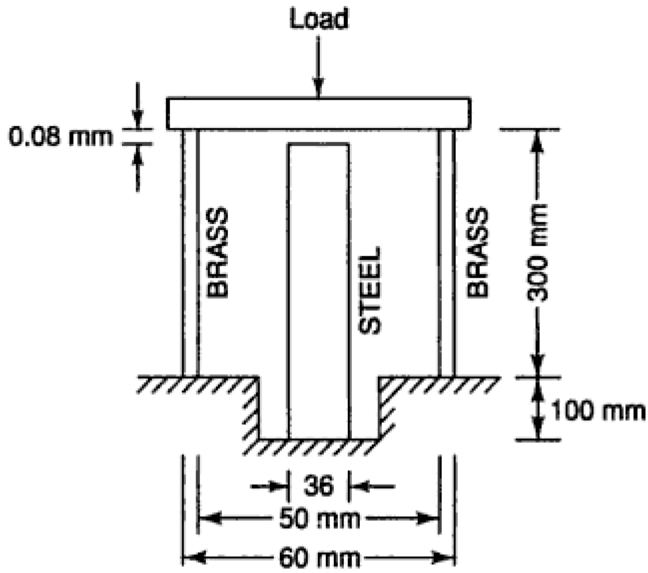


Fig. 1.17

Solution $$A_s = \frac{\pi}{4} \times 36^2 = 324\pi \text{ mm}^2$$

and $$A_b = \frac{\pi}{4} \times (60^2 - 50^2) = 275\pi \text{ mm}^2$$

(i) Let W_b be the load applied for the initial compression of the tube before the compression of the rod starts. Then

$$\Delta_b = \frac{\sigma_b L}{E} \quad \text{or} \quad 0.08 = \frac{\sigma_b \times 300}{105\,000}$$

or $$\sigma_b = 28 \text{ MPa and } W_b = 28 \times 275\pi = 24\,190 \text{ N}$$

But limiting value of stress in the brass = 75 MPa

∴ Maximum value of stress due to additional load can be = 75 – 28 = 47 MPa

Let W be the additional load to compress both, the tube and the bar. Let σ_s be the stress induced in the steel rod and σ_b the additional stress in the brass tube.

Equilibrium equation, $\sigma_s A_s + \sigma_b A_b = W$

Compatibility equation, $\Delta_s = \Delta_b$

or $\dfrac{\sigma_s L_s}{E_s} = \dfrac{\sigma_b L_b}{E_b}$ or $\sigma_s = \dfrac{L_b}{L_s}.\dfrac{E_s}{E_b}\sigma_b = \dfrac{300}{400}.\dfrac{210}{105}\sigma_b$ or $\sigma_s = 1.5\sigma_b$

Therefore the stress induced in the steel rod = 1.5 × 47 = 70.5 MPa

It is less than the permissible value of stress for steel.

Thus $W = p_s A_s + p_b A_b = 70.5 \times 324\pi + 47 \times 275\pi = 112\,365$ N

Total maximum load = 112 365 + 24 190 = 136 555 N or 136.555 kN

(ii) Let Δ be the shortening of the steel rod. This will also be the additional shortening of the brass tube. Then $\Delta_b + \Delta = \dfrac{\sigma_b . L_b}{E_b}$

or $$\sigma_b = \frac{105\,000}{300}(0.08 + \Delta) \text{ and } \sigma_s = \frac{210\,000}{400}.\Delta$$

Equating the stresses in the steel and the brass,

$$= \frac{105\,000}{300}(0.08 + \Delta) = \frac{210\,000}{400}\Delta \text{ or } 0.08 + \Delta = 1.5\,\Delta$$

or $0.5\,\Delta = 0.08$ or $\Delta = 0.16$ mm

Total shortening = 0.08 + 0.16 = 0.24 mm

Example 1.7 *Three wires of the same material and cross-section support a rigid bar which further supports a weight of 5 kN. The length of the wires is 5 m, 8 m and 6 m in order. The spacing between the wires is 2 m and the weight acts at 1.6 m from the first wire. Determine the load carried by each wire.*

Solution As the wires are of different lengths and the weight suspended is unsymmetrical, the bar will not remain horizontal but will be deformed as shown in Fig. 1.18.

Let P_1, P_2 and P_3 be the loads taken by the first, second and the third wire respectively.

Then $P_1 + P_2 + P_3 = P = 5000$ (i)

Taking moments about the first wire,

$2P_2 + 4P_3 = 1.6 \times 5000 = 8000$

or $P_2 = 4000 - 2P_3$ (ii)

Also, from symmetry,

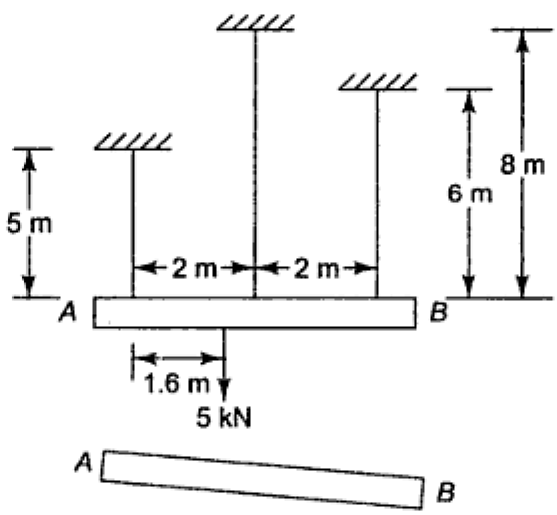


Fig. 1.18

$$\Delta_2 = \frac{\Delta_1 + \Delta_3}{2} \text{ or } 2\left(\frac{P_2 L_2}{AE}\right) = \frac{P_1 L_1}{AE} + \frac{P_3 L_3}{AE}$$

or $2P_2L_2 = P_1L_1 + P_3L_3$ or $2P_2 \times 8 = P_1 \times 5 + P_3 \times 6$

or $\qquad 16P_2 = 5P_1 + 6P_3$

or $\quad 16(4000 - 2P_3) = 5P_1 + 6P_3$ or $64\,000 - 32P_3 = 5P_1 + 6P_3$

or $\qquad 5P_1 = 64\,000 - 38P_3$ or $P_1 = 12\,800 - 7.6P_3$ (iii)

Inserting the values of P_1 and P_2 from (ii) and (iii) in (i),

$$12\,800 - 7.6P_3 + 4000 - 2P_3 + P_3 = 5000$$

or $\qquad 8.6P_3 = 11\,800$ or $P_3 = 1372$ N or 1.372 kN

$$P_2 = 4000 - 2P_3 = 4000 - 2 \times 1372 = 1256 \text{ N or } 1.256 \text{ kN}$$

$$P_1 = 12\,800 - 7.6 \times 1372 = 2373 \text{ N or } 2.373 \text{ kN}$$

Example 1.8 *A system of three bars supports a vertical load P as shown in Fig. 1.19. The outer bars are identical and of the same material whereas the inner bar is of different material. Determine the forces in the bars of the system.*

Solution Owing to symmetry, forces in the outer bars 1 and 3 will be equal. Let it be P_1 and the force in the inner bar P_2. The dotted lines show the deformed shape of the system under the load P.

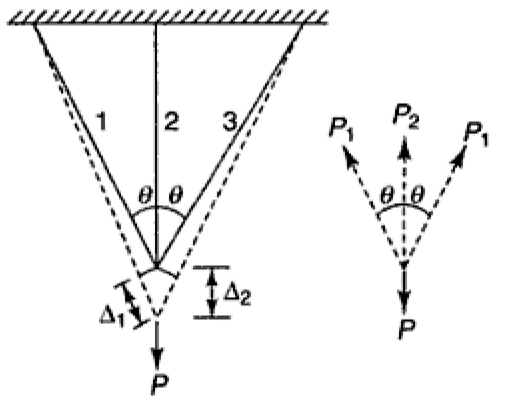


Fig. 1.19

From equilibrium equation,

$$2P_1 \cos\theta + P_2 = P \qquad \text{....(assuming negligible change in } \theta) \qquad \text{(i)}$$

From compatibility equation,

$$\Delta_1 = \Delta_2 \cos\theta \quad \text{or} \quad \frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} \cos\theta$$

or
$$P_1 = \frac{A_1 E_1 P_2 L_2}{A_2 E_2 L_1} \cos\theta = \frac{A_1 E_1 P_2 (L_1 \cos\theta)}{A_2 E_2 L_1} \cos\theta = \frac{A_1 E_1 P_2}{A_2 E_2} \cos^2\theta \qquad \text{(ii)}$$

Substituting this value of P_1 in (i),

$$2\frac{A_1 E_1 P_2}{A_2 E_2} \cos^3\theta + P_2 = P \quad \text{or} \quad P_2 = \frac{P}{1 + \dfrac{2A_1 E_1}{A_2 E_2} \cos^3\theta}$$

From (ii),
$$P_1 = \frac{A_1 E_1}{A_2 E_2} \cdot \frac{P}{1 + \dfrac{2A_1 E_1}{A_2 E_2} \cos^3\theta} . \cos^2\theta$$

$$= P\cos^2\theta \frac{1}{\left(\dfrac{A_2 E_2}{A_1 E_1}\right)} \cdot \frac{1}{\left(1 + \dfrac{2A_1 E_1}{A_2 E_2} \cos^3\theta\right)} = \frac{P\cos^2\theta}{\left(\dfrac{A_2 E_2}{A_1 E_1} + 2\cos^3\theta\right)}$$

Example 1.9 *Figure 1.20 shows a horizontal bar supported by two suspended vertical wires fixed to a rigid support. A load W is attached to the bar. The left hand side wire is of copper with a diameter of 5 mm and the right hand side wire is of steel of 3 mm diameter. The length of both the wires is 4 m initially. Find the position of the weight on the bar so that both the wires extend by the same amount.*

Also, calculate the load, stresses and the elongation of each wire if W = 1000 N. Neglect the weight of the bar and take E_s = 210 GPa and E_c = 120 GPa.

Solution $A_c = \frac{\pi}{4}(5)^2 = 6.25\pi \text{ mm}^2$

and $A_s = \frac{\pi}{4}(3)^2 = 2.25\pi \text{ mm}^2$

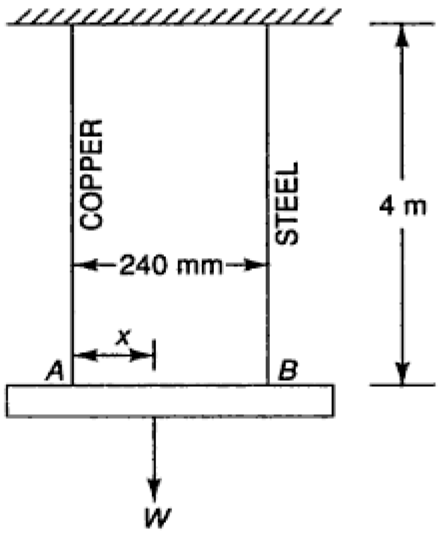


Fig. 1.20

Let the load W be placed at a distance x from the copper wire and P_s and P_c the forces in steel and copper wires respectively.

Then taking moments about A, $240\, P_s = W.x$

or $P_s = \dfrac{W.x}{240}$ (i)

Taking moments about B, $240\, P_c = W(240 - x)$

or $P_c = \dfrac{W.(240 - x)}{240}$ (ii)

Dividing (ii) by (i), $\dfrac{P_c}{p_s} = \dfrac{240 - x}{x}$ (*iii*)

As both the wires extend by the same amount, $\Delta_c = \Delta_s$

or $$\frac{P_c L_c}{A_c E_c} = \frac{P_s L_s}{A_s E_s} \text{ or } \frac{P_c}{p_s} = \frac{A_c}{A_s}.\frac{E_c}{E_s} \quad \ldots\ldots \quad (\because L_c = L_s)$$

$$= \frac{6.25\pi}{2.25\pi}.\frac{120\ 000}{210\ 000} = 1.587 \quad \text{(iv)}$$

From (iii) and (iv), $\dfrac{240 - x}{x} = 1.587$ or $x = 92.77$ mm

Numerical:

$$P_c = \frac{W(240 - x)}{240} = \frac{1000 \times (240 - 92.77)}{240} = 613.46 \text{ N}$$

$$P_s = \frac{Wx}{240} = \frac{1000 \times 92.77}{240} = 386.54 \text{ N}$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{613.46}{6.25\pi} = 31.24 \text{ MPa}$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{386.54}{2.25\,\pi} = 54.68 \text{ MPa}$$

$$\Delta = \frac{\sigma_c.L}{E_c} = \frac{31.24 \times 4000}{120\ 000} = 1.041 \text{ mm}$$

Example 1.10 *Three identical pin-connected bars support a load P as shown in Fig. 1.21. All the bars are of the same area of cross-section and same length. Determine*

(i) the force in each bar

(ii) the vertical displacement of the point where the load is applied

Neglect the possibility of lateral buckling of the bars.

Solution

(i) The dotted lines show the deformed shape of the structure. Assuming that there is negligible change in the angles after the deforming of the bars.

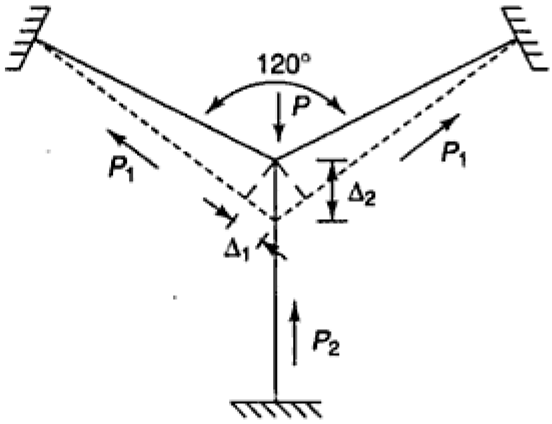


Fig. 1.21

Equilibrium equation

$$2P_1 \cos 60^\circ + P_2 = P$$

or $\qquad P_1 = P - P_2 \qquad$ (i)

Compatibility equation, $\Delta_1 = \Delta_2 \cos 60^\circ$

or $\qquad \frac{P_1L}{AE} = \frac{P_2L}{AE}\cos 60^\circ \text{ or } P_1 = \frac{P_2}{2} \qquad$ (ii)

From (i) and (ii), $\frac{P_2}{2} = P - P_2$ or $P_2 = \frac{2P}{3}$ and $P_1 = P/3$

(ii) Vertical displacement of the joint, $\Delta_2 = \frac{P_2L}{AE} = \frac{2PL}{3AE}$

Example 1.11 *A bar system is loaded as shown in Fig. 1.22. Determine*

(i) the reaction of the lower support, and (ii) the stresses in the bars.

Take E = 205 GPa

Solution

(i) When the load is applied and the support touches it, the reactions of both the supports will be upward since the load is downward.

Let R_1 = reaction of the upper support

R_2 = reaction of the lower support

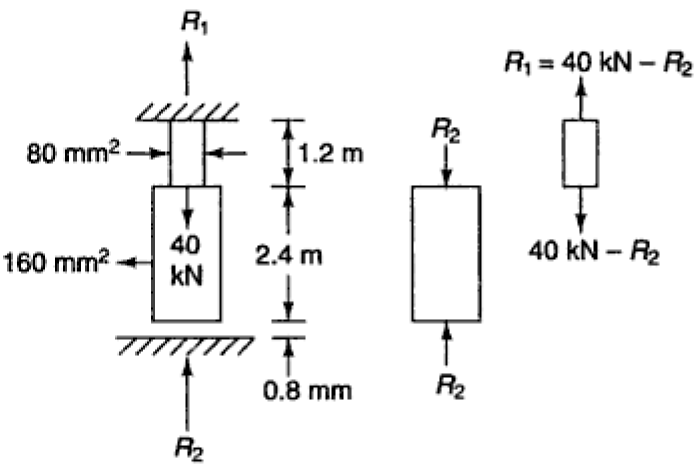


Fig. 1.22

Then $R_1 + R_2 = 40\,000$

or $\qquad R_1 = 40\,000 - R_2$

The free-body diagrams of the two portions of the bar system is shown in the figure. It is clear that the upper portion is in tension whereas the lower portion in compression.

Elongation of the upper portion, $\Delta_1 = \dfrac{P_1L_1}{A_1E} = \dfrac{(40\,000 - R_2)\times 1200}{80\times 205\,000}$

Shortening of the lower portion, $\Delta_2 = \dfrac{P_2L_2}{A_2E} = \dfrac{R_2\times 2400}{160\times 205\,000}$

From compatibility equation,

Elongation of upper portion – shortening of lower portion = net elongation = 0.8 mm

or $$\frac{(40\,000 - R_2)\times 1200}{80\times 205\,000} - \frac{R_2\times 2400}{160\times 205\,000} = 0.8$$

$$(40\,000 - R_2)\times 15 - 15R_2 = 0.8\times 205\,000$$

or $40\,000 - 2R_2 = 10\,933$ or $R_2 = 14\,533$ N

and $R_1 = 40\,000 - 14\,533 = 25\,467$ N

(ii) $\sigma_1 = 25\,467/80 = 318.3$ MPa (tensile)

$\sigma_2 = 14\,533/160 = 90.8$ MPa (compressive)

Example 1.12 *A rigid horizontal bar AB hinged at A is supported by a 1.2-m long steel rod and a 2.4-m long bronze rod, both rigidly fixed at the upper ends (Fig. 1.23). A load of 48 kN is applied at a point 3.2 m from the hinge point A. The areas of cross-section of the steel and bronze rods are 850 mm² and 650 mm² respectively. Find*

(i) stress in each rod (ii) reaction at the pivot point.

E_s = 205 GPa and E_b = 82 GPa

Solution Refer Fig. 1.23.

(i) Let P_s and P_b be the forces in the steel and bronze wires respectively as the load is applied. Taking moments about the pivot point,

$P_s\times 800 + P_b\times 2400 - 48\,000\times 3200 = 0$

$P_s + 3P_b = 192\,000$ (i)

From compatibility equation,

$$\frac{\Delta_s}{\Delta_b} = \frac{800}{2400} = \frac{1}{3} \text{ or } \frac{\Delta_s}{1} = \frac{\Delta_b}{3}$$

or $$\frac{P_sL_s}{A_sE_s} = \frac{1}{3}\frac{P_bL_b}{A_bE_b}$$

or $$\frac{P_s\times 1200}{850\times 205\,000} = \frac{1}{3}\times\frac{P_b\times 2400}{650\times 82\,000}$$

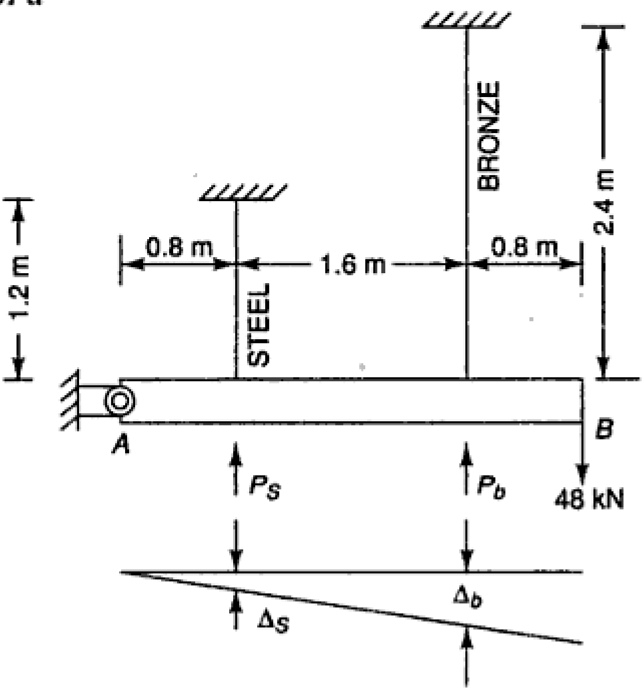


Fig. 1.23

$P_s = 2.179\,P_b$ (ii)

From (i) and (ii), $2.179\,P_b + 3P_b = 192\,000$

or $P_b = 37\,073$ N

$$P_s = 192\,000 - 3 \times 37\,073 = 80\,781 \text{ N}$$

$$\sigma_b = \frac{37\,073}{650} = 57.04 \text{ MPa} \quad \text{and} \quad \sigma_s = \frac{80\,781}{850} = 95.04 \text{ MPa}$$

(ii) The reaction at the pivot can be found from force equation, let it be downwards,

$$P_s + P_b - R_a = 48\,000$$

$$R_a = 80\,781 + 37\,073 - 48\,000 = 69\,854 \text{ N or } 69.854 \text{ kN}$$

Thus the assumed direction is correct.

Example 1.13 *A rigid bar AB is to be suspended from three steel rods as shown in Fig. 1.24a. The lengths of the outer rods are 1.5 m each whereas the length of the middle rod is shortened than these by an amount of 0.8 mm. The area of cross-section of all the rods is the same and is equal to 1600 mm². Determine the stresses in the rods after the assembly of the structure.* E_s *= 205 GPa.*

Solution The position of the rigid bar after the assembly is shown in Fig. 1.24*b*. It is raised upward by amounts Δ_1, Δ_2 and Δ_3 at the rod positions 1, 2 and 3 respectively. Thus the rods 1 and 3 are shortened by amounts Δ_1 and Δ_3 respectively whereas rod 2 is elongated by an amount $(0.8 - \Delta_2)$.

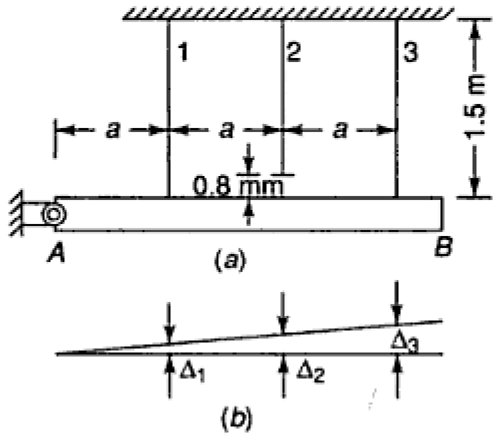


Fig. 1.24

We have, $\dfrac{\Delta_3}{\Delta_1} = \dfrac{3a}{a}$

or $\dfrac{P_3L_3/A_3E_3}{P_1L_1/A_1E_1} = 3$

As $L_3 = L_1$, $A_3 = A_1$ and $E_3 = E_1$

$\therefore$ $P_3 = 3P_1$ (i)

Also, $\dfrac{\Delta_2}{\Delta_1} = \dfrac{2a}{a}$ or $\dfrac{0.8 - P_2L_2 / A_2E_2}{P_1L_1 / A_1E_1} = 2$

The length of the rod 2 is shorter by 0.8 mm. However, to find the elongation of the rod, this may be ignored as its effect will be negligible and the length of rod 2 can be taken equal to that of rod 1.

Thus $L_2 = L_1$

Also, $A_2 = A_1$ and $E_2 = E_1$

$\therefore$ $\dfrac{0.8 - P_2L_1 / A_1E_1}{P_1L_1 / A_1E_1} = 2$ or $\dfrac{0.8 \times A_1E_1}{P_1L_1} - \dfrac{P_2}{P_1} = 2$

or $\dfrac{0.8 \times 1600 \times 205\,000}{P_1 \times 1500} - \dfrac{P_2}{P_1} = 2$ or $174\,933 - P_2 = 2P_1$

or $2P_1 + P_2 = 174\,933$ (ii)

Taking moments about A, $P_1.a + P_3.3a = P_2.2a$

or $P_1 + 3P_3 = 2P_2$ (iii)

Solving (i), (ii) and (iii),

From (i) and (iii), $P_1 + 3 \times 3P_1 = 2P_2$ or $P_2 = 5P_1$ (*iv*)

From (ii) and (iv), $2P_1 + 5P_1 = 174\,933$ or $P_1 = 24\,990$ N

$$P_2 = 24\,990 \times 5 = 124\,952 \text{ N}$$

$$P_3 = 24\,990 \times 3 = 74\,971 \text{ N}$$

$$\sigma_1 = 24\,990/1600 = 15.62 \text{ MPa (compressive)}$$

$$\sigma_2 = 15.62 \times 5 = 78.1 \text{ MPa (tensile)}$$

$$\sigma_3 = 15.62 \times 3 = 46.86 \text{ MPa (compressive)}$$

1.12 TEMPERATURE STRESSES

The length of a material which undergoes a change in temperature also changes and if the material is free to do so, no stresses are developed in the material. However, if the material is constrained, stresses are developed in the material which are known as *temperature stresses*.

Consider a bar of length L. If its temperature is increased through $t°$, its length is increased by an amount $L.\alpha.t$, where α is the coefficient of thermal expansion. But if the bar is constrained and is prevented from expansion, the temperature stress is induced in the material which is given by

$$E = \frac{\text{temperature stress}}{\text{temperature strain}} = \frac{\sigma}{L\alpha t/L}$$

or $$\sigma = \alpha\, t\, E \tag{1.14}$$

or $$\sigma = \alpha\, t\, \sigma/\varepsilon$$

or temperature strain, $\varepsilon = \alpha.t$ (1.15)

Compound Sections

Consider a copper rod enclosed in a steel tube as shown in Fig. 1.25 rigidly joined at each end. Now, if the temperature is increased by $t°$, the copper rod would tend to expand more as compared to steel tube. As the two are joined together, the copper is prevented its full expansion and is put in compression. The final position of the compound bar will be as shown in the figure.

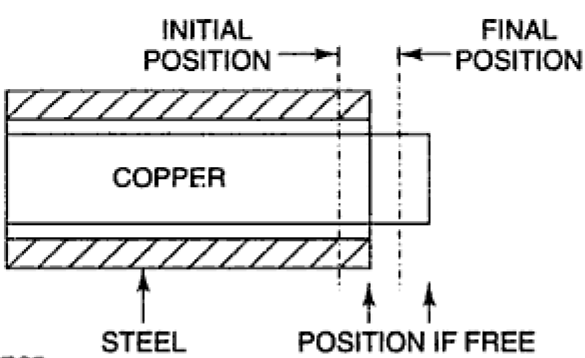


Fig. 1.25

Let σ_s = tensile stress in steel

σ_c = compressive stress in copper

A_s = cross-sectional area of steel tube

A_c = cross-sectional area of copper rod

From equilibrium equation

Tensile force in steel = compressive force in copper

$$\sigma_s . A_s = \sigma_c . A_c \tag{1.16}$$

or $$\varepsilon_s . E_s . A_s = \varepsilon_c . E_c . A_c$$

Compatibility equation:

Let α_s = coefficient of thermal expansion in steel

α_c = coefficient of thermal expansion in copper

Now Elongation of steel tube (due to temperature + due to tensile stress)
= Elongation of copper rod (due to temperature – due to compressive stress)

or Temperature strain of steel + tensile strain
= Temperature strain of copper – compressive strain

$$\alpha_s t + \sigma_s / E_s = \alpha_c t - \sigma_c / E_c$$

or $$\alpha_s t + \varepsilon_s = \alpha_c t - \varepsilon_c$$

or $$\varepsilon_s + \varepsilon_c = (\alpha_c - \alpha_s) t \tag{1.17}$$

Equations (1.16) and (1.17) are sufficient to solve the problems.

Example 1.14 *Two parallel walls 8 m apart are to be stayed together by a steel rod of 30-mm diameter with the help of washers and nuts at the ends. The steel rod is passed through the metal plates and is heated. When its temperature is raised to 90°C, the nuts are tightened. Determine the pull in the bar when it is cooled to 24°C if*

(i) the ends do not yield (ii) the total yielding at the ends is 2 mm.

E = 205 GPa and coefficient of thermal expansion of steel, $\alpha_s = 11 \times 10^{-6}/°C$.

Solution

$$A = \frac{\pi}{4}(30)^2 = 225\pi \text{ mm}^2$$

(i) Pull in the bar, $P = \sigma . A = \alpha t E A$

$$= 11 \times 10^{-6} \times (90 - 24) \times 205\,000 \times 225\,\pi = 105\,202 \text{ N}$$

(ii) When the yield at the ends is 2 mm,

$$\Delta = (\alpha L t - 2) = \frac{PL}{AE}$$

or $$P = \alpha\, t\, A\, E - \frac{2AE}{L} = 105\,202 - \frac{2 \times 225\pi \times 205\,000}{8000} = 105\,202 - 36\,227$$

$$= 68\,975 \text{ N} \quad \text{or} \quad 68.975 \text{ kN}$$

Example 1.15 *A composite bar made up of copper, steel and brass is rigidly attached to the end supports as shown in Fig. 1.26. Determine the stresses in the three portions of the bar when the temperature of the composite system is raised by 70°C when*

(i) the supports are rigid (ii) the supports yield by 0.6 mm.

$$E_c = 100 \text{ GPa}; \; E_s = 205 \text{ GPa}; \; E_b = 95 \text{ GPa}$$

$$\alpha_c = 18 \times 10^{-6}/°C; \; \alpha_s = 11 \times 10^{-6}/°C; \; \alpha_b = 19 \times 10^{-6}/°C$$

Solution

$$A_c = (\pi/4)50^2 = 625\ \pi\ \text{mm}^2;$$
$$A_s = (\pi/4)40^2 = 400\ \pi\ \text{mm}^2;\ A_b = (\pi/4)60^2 = 900\ \pi\ \text{mm}^2$$

(i) When the temperature is raised, each portion tends to elongate which is resisted by the rigid supports and the compressive stresses are developed in each portion. However, the forces so developed in each portion are equal,

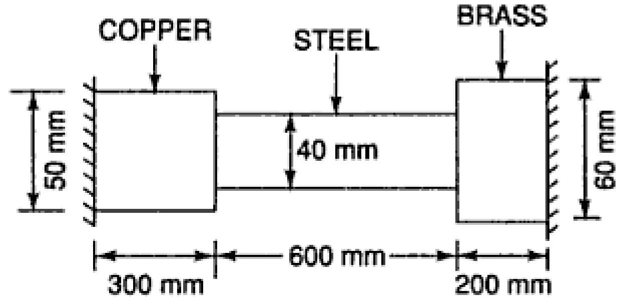


Fig. 1.26

i.e. $\sigma_c.A_c = \sigma_s.A_s = \sigma_b.A_b$ or $\sigma_c = \dfrac{A_b}{A_c}\sigma_b = \dfrac{900\pi}{625\pi}\times\sigma_b = 1.44\sigma_b$

and $\sigma_s = \dfrac{A_b}{A_s}\sigma_b = \dfrac{900\pi}{400\pi}\times\sigma_b = 2.25\sigma_b$

Elongation in the absence of supports, $\Delta = \Delta_c + \Delta_s + \Delta_b$

$$= \alpha_c L_c t_c + \alpha_s L_s t_s + \alpha_b L_b t_b$$
$$= 18\times10^{-6}\times300\times70 + 11\times10^{-6}\times600\times70 + 19\times10^{-6}\times200\times70$$
$$= 70\times10^{-6}(5400 + 6600 + 3800) = 1.106\ \text{mm}$$

Also from stress considerations, $\Delta = \dfrac{\sigma_c L_c}{E_c} + \dfrac{\sigma_s L_s}{E_s} + \dfrac{\sigma_b L_b}{E_b}$

Thus, $$\frac{1.44\sigma_b\times300}{100\ 000} + \frac{2.25\sigma_b\times600}{205\ 000} + \frac{\sigma_b\times200}{95\ 000} = 1.106$$

or $$(0.004\ 32 + 0.006\ 59 + 0.002\ 11)\ \sigma_b = 1.106$$

$$0.01302\ \sigma_b = 1.106$$
$$\sigma_b = 84.95\ \text{MPa}$$
$$\sigma_c = 84.95\times1.44 = 122.33\ \text{MPa}$$
$$\sigma_s = 84.95\times2.25 = 191.13\ \text{MPa}$$

(ii) When the supports yield by 0.6 mm,

$$0.0132\ \sigma_b = 1.106 - 0.6 = 0.506$$
$$\sigma_b = 38.33\ \text{MPa}$$
$$\sigma_c = 38.33\times1.44 = 55.20\ \text{MPa}$$
$$\sigma_s = 38.33\times2.25 = 86.24\ \text{MPa}$$

Example 1.16 *A steel tube of 35-mm outer diameter and 30-mm inner diameter encloses a gun metal rod of 25-mm diameter and is rigidly joined at each end. If at a temperature of 40°C there is no longitudinal stress, determine the stresses developed in the rod and the tube when the temperature of the assembly is raised to 240°C.*

Coefficient of thermal expansion of steel = 11 × 10^{-6} /°C.

Coefficient of thermal expansion of gun metal = 18 × 10⁻⁶/°C.
Young's modulus for steel = 205 GPa
Young's modulus for gun metal = 91.5 GPa
Also find the increase in length if the original length of the assembly is 1 m.

Solution Refer Fig. 1.27,

$$A_s = \frac{\pi}{4}(35^2 - 30^2) = 255.25 \text{ mm}^2$$

and $A_g = \frac{\pi}{4} \times 25^2 = 490.87 \text{ mm}^2$

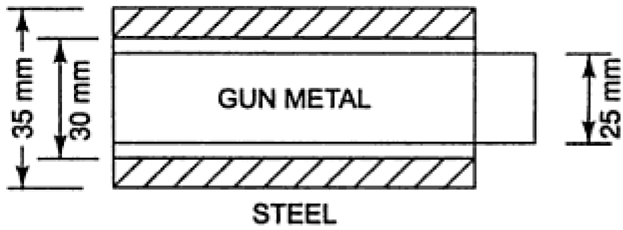


Fig. 1.27

As the coefficient of expansion of the gun metal is more as compared to that of steel, the final expansion will be less than the free expansion of gun metal due to temperature rise and thus compressive stresses will be developed in the gun metal rod. In a similar way, as the coefficient of expansion of the steel is less, the final expansion will be more than the free expansion of steel due to temperature rise and thus it will have tensile stresses.

Temperature strain of steel + tensile strain

= Temperature strain of copper – compressive strain

i.e. $$\alpha_s t + \frac{\sigma_s}{E_s} = \alpha_g t - \frac{\sigma_g}{E_g} \quad \text{or} \quad \alpha_s t + \frac{P}{A_s E_s} = \alpha_g t - \frac{P}{A_g E_g}$$

or $$P\left(\frac{1}{A_s E_s} + \frac{1}{A_g E_g}\right) = t(\alpha_g - \alpha_s)$$

or $$P = \frac{t(\alpha_g - \alpha_s)}{\frac{1}{A_s E_s} + \frac{1}{A_g E_g}} = \frac{(240 - 40)(18 - 11) \times 10^{-6}}{\frac{1}{255.25 \times 205\,000} + \frac{1}{490.87 \times 91\,500}}$$

$$= \frac{1400 \times 10^{-6}}{19.11 \times 10^{-9} + 22.26 \times 10^{-9}} = 33841 \text{ N}$$

$$\sigma_s = \frac{33841}{255.25} = 132.6 \text{ MPa and } \sigma_g = \frac{33\,841}{490.87} = 68.94 \text{ MPa}$$

Increase in length of assembly

= Elongation of steel tube (due to temperature + due to tensile stress)

= Elongation of copper rod (due to temperature – due to compressive stress)

Using the first equation, Increase in length

$$= \alpha_s L t + \frac{\sigma_s L}{E_s} = L\left(\alpha_s t + \frac{\sigma_s}{E_s}\right)$$

$$= 1000\left(11 \times 10^{-6} \times 200 + \frac{132.6}{205\,000}\right) = 2.847 \text{ mm}$$

Example 1.17 *Rails are laid such that there is no stress in them at 24° C. If the rails are 32-m long, determine*

(i) the stress in the rails at 80°C, when there is no allowance for expansion.

(ii) the stress in the rails at 80°C, when there is an expansion allowance of 8 mm per rail.

(iii) The expansion allowance for no stress in the rails at 80°C.

(iv) The maximum temperature for no stress in the rails when expansion allowance is 8 mm.

Coefficient of linear expansion, $\alpha = 11 \times 10^{-6}/°C$ *and* $E = 205$ *GPa*

Solution Change in temperature = 80° – 24° = 56°

(i) When there is no allowance for expansion,

$$\sigma = \alpha t E = 11 \times 10^{-6} \times 56 \times 205\,000 = 126.28 \text{ MPa}$$

(ii) When there is an expansion allowance of 8 mm, $\Delta = \alpha L t - 8 = \dfrac{\sigma L}{E}$

or $$11 \times 10^{-6} \times 32\,000 \times 56 - 8 = \frac{\sigma \times 32\,000}{205\,000}$$

or $$19.712 - 8 = 0.1561\,\sigma \quad \text{or} \quad \sigma = 75.03 \text{ MPa}$$

(iii) If stresses are to be zero, the expansion allowance

$$\Delta = \alpha L t = 11 \times 10^{-6} \times 32\,000 \times 56 = 19.71 \text{ mm}$$

(iv) For no stress in the rails when expansion allowance is 8 mm.

$$8 = \alpha L t$$

or $$8 = 11 \times 10^{-6} \times 32\,000 \times t \qquad \text{or} \qquad t = 22.73°\text{C}$$

Example 1.18 *A steel rod of 16-mm diameter and 3-m length passes through a copper tube of 50-mm external and 40-mm internal diameter and of the same length. The tube is closed at each end with the help of 30 mm thick steel plates which are tightened by nuts till the length of the copper tube is reduced by 0.6 mm. The temperature of the whole assembly is then raised by 56°C. Determine the stresses in the steel and copper before and after the rise of temperature. Assume that the thickness of the steel plates at the ends do not change during tightening of the nuts.*

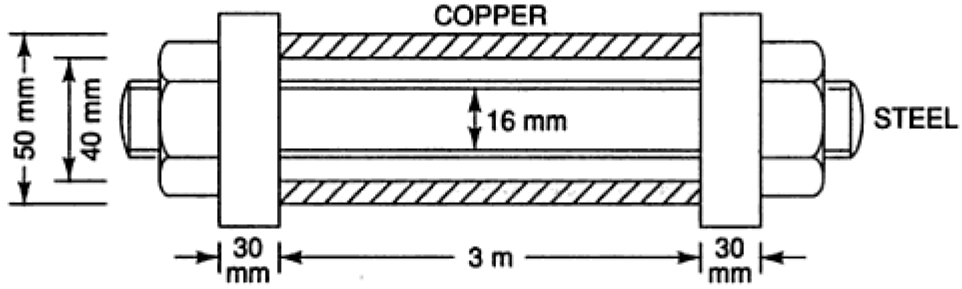


Fig. 1.28

$E_s = 210$ *GPa;* $E_c = 100$ *GPa;*

$\alpha_s = 12 \times 10^{-6}/°C;\ \alpha_c = 17 \times 10^{-6}/°C$

Solution Refer Fig. 1.28.

$$A_s = (\pi/4)16^2 = 64\,\pi\text{ mm}^2;$$
$$A_c = (\pi/4)\,[50^2 - 40^2] = 225\,\pi\text{ mm}^2$$

Stresses due to tightening of the nuts

As $\Delta = \dfrac{\sigma_c L}{E_c}$ $\therefore$ $0.6 = \dfrac{\sigma_c \times 3000}{100\,000}$ or $\sigma_c = 20$ MPa (compressive)

and as the force in the rod and the tube is the same, $\sigma_s.A_s = \sigma_c.A_c$

or $\sigma_s \times 64\pi = 20 \times 225\,\pi$ or $\sigma_s = 70.3$ MPa (tensile)

Stresses due to temperature rise

As the coefficient of expansion of copper is more than that of steel, it expands more. Thus compressive stress is induced in the copper tube and tensile in the steel rod.

As $\sigma_s.A_s = \sigma_c.A_c$

$\therefore$ $\sigma_s = (A_c/A_s)\,\sigma_c = (225/64)\,\sigma_c = 3.516\,\sigma_c$

Now, from compatibility equation,

Temperature strain of steel + tensile strain of steel

= Temperature strain of copper – compressive strain of copper

i.e. $$\alpha_s.L_s t + \frac{\sigma_s L_s}{E_s} = \alpha_c L_c t - \frac{\sigma_c L_c}{E_c}.$$

$$12 \times 10^{-6} \times (3000 + 60) \times 56 + \frac{3.516\,\sigma_c \times 3060}{210\,000}$$
$$= 17 \times 10^{-6} \times 3000 \times 56 - \frac{\sigma_c \times 3000}{100\,000}$$

or $2.056 + 0.051\sigma_c = 2.856 - 0.03\,\sigma_c$

or $0.081\,\sigma_c = 0.8$ or $\sigma_c = 9.87$ MPa

and $\sigma_s = 3.516 \times \sigma_c = 3.516 \times 9.87 = 34.7$ MPa

Final stresses

$\sigma_c = 20 + 9.87 = 29.87$ MPa (compressive)

and $\sigma_s = 70.3 + 34.6 = 104.9$ MPa (tensile)

Example 1.19 *A steel rod of 30-mm diameter is enclosed in a brass tube of 42-mm external diameter and 32-mm internal diameter. Each is 360 mm long and the assembly is rigidly held between two stops 360 mm apart. The temperature of the assembly is then raised by 50°C. Determine*

(i) stresses in the tube and the rod

(ii) stresses in the tube and the rod if the stops yields by 0.15 mm

(iii) yield of the stops if the force at the stops is limited to 60 kN

$E_s = 205$ GPa; $E_b = 90$ GPa; $\alpha_s = 11 \times 10^{-6}/°C$; $\alpha_b = 19 \times 10^{-6}/°C$

Solution Refer Fig. 1.29.

$A_s = (\pi/4)\,30^2 = 225\,\pi\text{ mm}^2;$

$A_b = (\pi/4)\ [42^2 - 32^2]$
$= 185\pi\ \text{mm}^2$

(i) When the temperature is raised by 50°,

Stress in the steel rod = $\alpha_s\, t\, E_s$
$= 11 \times 10^{-6} \times 50 \times 205\ 000$
$= 112.75$ MPa (compressive)

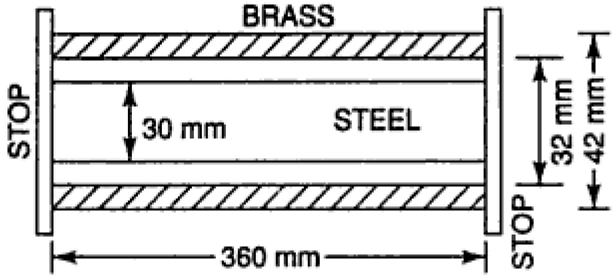


Fig. 1.29

Stress in the brass tube = $\alpha_b\, t\, E_b = 19 \times 10^{-6} \times 50 \times 90\ 000$
$= 85.5$ MPa (compressive)

(*ii*) If the stops yields by 0.15 mm, $\Delta_s = (\alpha_s L t - 0.15) = \dfrac{\sigma_s L}{E_s}$

or $\quad \sigma_s = \alpha_s t E_s - \dfrac{0.15 E_s}{L} = 112.75 - \dfrac{0.15 \times 205\ 000}{360}$
$= 112.75 - 85.42 = 27.33$ MPa (compressive)

and $\quad \Delta_b = (\alpha_b L t - 0.15) = \dfrac{\sigma_b L}{E_b}$

or $\quad \sigma_b = \alpha_b t E_b - \dfrac{0.15 E_b}{L} = 85.5 - \dfrac{0.15 \times 90\ 000}{360}$
$= 85.5 - 37.5 = 48$ MPa (compressive)

(iii) When the force at the stops is limited to 60 kN, let the yield of the stops be δ,

Then $\quad \Delta_s = (\alpha_s L t - \delta) = \dfrac{\sigma_s L}{E_s}$

or $\quad \sigma_s = \alpha_s t E_s - \dfrac{\delta E_s}{L} = 112.75 - \dfrac{\delta \times 205\ 000}{360} = 112.75 - 569.44\,\delta$

and $\quad \Delta_b = (\alpha_b L t - \delta) = \dfrac{\sigma_b L}{E}$

or $\quad \sigma_b = \alpha_b t E_b - \dfrac{\delta E_b}{L} = 85.5 - \dfrac{\delta \times 90\ 000}{360}$
$= 85.5 - 250\,\delta$

Now, Force exerted by steel rod + Force exerted by brass tube = total force on the stops

$$\sigma_s . A_s + \sigma_b . A_b = P$$
$$(112.75 - 569.44\,\delta) \times 225\,\pi + (85.5 - 250\,\delta) \times 185\,\pi = 60\ 000$$
$$79\ 698 - 402\ 513\,\delta + 49\ 692 - 145\ 299\,\delta = 60\ 000$$
$$547\ 812\,\delta = 69\ 390$$
$$\delta = 0.127 \text{ mm}$$

Example 1.20 *A rigid block AB weighing 180 kN is supported by three rods symmetrically placed as shown in Fig. 1.30. Before attaching the weight, the lower ends of the rods are set at the same level. The areas of cross-section of the steel and copper rods are 800 mm² and 1350 mm² respectively. Determine* *(i) the stresses in the rods, if the temperature is raised by 25°*

(ii) the stresses in the rods, if the temperature is raised by 50°

(iii) the temperature rise for no stress in the copper rod.

E_c = 95 GPa; $\alpha_c = 18 \times 10^{-6}$/°C;

E_s = 205 GPa; $\alpha_s = 11 \times 10^{-6}$/°C

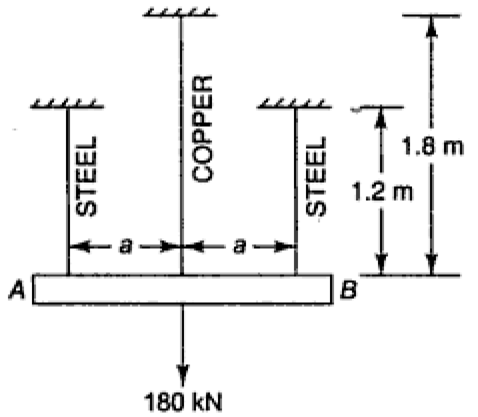


Fig. 1.30

Solution Considering the increase in temperature alone (neglecting the weight of the block), the elongation of copper rod is more as compared to steel rods. On the other hand, if the temperature does not change, there is elongation of all rods and there is tensile stress in all the rods.

Total elongation of each rod is the sum of elongations due to temperature and due to weight. As the block is rigid, it will remain horizontal under all conditions. Thus the total elongation of each rod is the same.

(i) Assume the stress in the copper rod to be compressive, i.e. the force acting upwards.

Then $$\alpha_s L_s t + \frac{P_s L_s}{A_s E_s} = \alpha_c L_c t - \frac{P_c L_c}{A_c E_c}$$

$$11 \times 10^{-6} \times 1200 \times 25 + \frac{P_s \times 1200}{800 \times 205\,000}$$

$$= 18 \times 10^{-6} \times 1800 \times 25 - \frac{P_c \times 1800}{1350 \times 95\,000}$$

$$330\,000 + 7.317 P_s = 810\,000 - 14.035 P_c$$

$$P_s = 65\,601 - 1.918 P_c \qquad \text{(i)}$$

From equilibrium equation

$$2P_s - P_c = 180\,000$$

or $$P_s - 0.5P_c = 90\,000$$

$$65\,601 - 1.918 P_c - 0.5 P_c = 90\,000 \qquad \text{[from (i)]}$$

or $$2.418\, P_c = -\,24\,399$$

or $$P_c = -\,10\,090 \text{ N (compressive)}$$

and $$P_s = 90\,000 + 0.5 \times (-\,10\,090) = 84\,955 \text{ (tensile)}$$

$$\sigma_c = -\frac{10\,090}{1350} = -\,7.474 \text{ MPa}$$

This shows that the stress in the copper rod is opposite of what was assumed i.e. tensile and not compressive.

$$\sigma_s = \frac{84\,955}{800} = 106.19 \text{ MPa (tensile)}$$

(ii) If the temperature is raised by 50°,

$$11 \times 10^{-6} \times 1200 \times 50 + 7.317 \times 10^{-6} P_s$$
$$= 18 \times 10^{-6} \times 1800 \times 50 - 14.035 \times 10^{-6} P_c$$
$$660\,000 + 7.317\,P_s = 1\,620\,000 - 14.035 P_c$$
$$P_s = 131\,201 - 1.918\,P_c$$

From equilibrium equation

$$2P_s - P_c = 180\,000 \quad \text{or} \quad P_s - 0.5P_c = 90\,000$$

or $131\,201 - 1.918\,P_c = 90\,000$

or $P_c = 17\,039$ N (compressive)

$$\sigma_c = 17\,039/1350 = 12.62 \text{ MPa (compressive)}$$
$$P_s = 90\,000 + 0.5 \times 17\,039 = 98\,520 \text{ N}$$
$$\sigma_s = 98\,520/800 = 123.1 \text{ MPa (tensile)}$$

(iii) As there is to be no stress and hence no load on the copper rod, $\sigma_c = 0$

Hence load in each rod = 180 000/2 = 90 000 N

$$11 \times 10^{-6} \times 1200 \times t + 7.317 \times 10^{-6} P_s$$
$$= 18 \times 10^{-6} \times 1800 \times t - 0$$
$$13\,200\,t + 7.317\,P_s = 32\,400t$$
$$19\,200\,t = 7.317 \times 90\,000$$
$$t = 34.3°$$

1.13 SHRINKING ON

A thin tyre of steel or of any other metal can be shrunk on to wheels of slightly larger diameter by heating the tyre to a certain degree which increases its diameter. When the tyre has been mounted and the temperature falls to the normal temperature, the steel tyre tends to come to its original diameter and thus tensile (hoop) stress is set up in the tangential direction.

As shown in Fig. 1.31, let d and D be the diameters of the steel tyre and of the wheel on which the steel tyre is to be mounted (Fig. 1.31), then

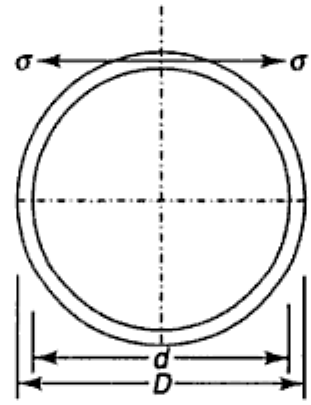


Fig. 1.31

The strain, $$\varepsilon = \frac{\pi D - \pi d}{\pi d} = \frac{D - d}{d}$$

Circumferential tensile stress or hoop stress $= \varepsilon . E = \left(\dfrac{D-d}{d} \right) E$ (1.18)

Example 1.21 *A thin tyre of steel is to be mounted on to a rigid wheel of 1.2-m diameter. Determine the internal diameter of the tyre if the hoop stress is limited to 120 MPa.*

Also determine the least temperature to which the tyre should be heated so that it can be slipped on to the wheel.

$$E_s = 210 \text{ GPa and } \alpha_s = 11 \times 10^{-6}/°C$$

Solution

$$\text{Tensile strain, } \varepsilon = \frac{\pi D - \pi d}{\pi d} = \frac{D-d}{d} = \left(\frac{\sigma}{E}\right)$$

or $$\frac{D}{d} = \left(\frac{\sigma}{E}\right) + 1 = \frac{\sigma + E}{E} \quad \text{or} \quad \frac{d}{D} = \frac{E}{\sigma + E}$$

or $$d = \frac{DE}{\sigma + E} = \frac{1200 \times 210\,000}{120 + 210\,000} = 1199.31 \text{ mm or } 1.19931 \text{ m}$$

Increase in the circumferential length = $\pi (D-d)$

Thus $\alpha L t = \pi (D-d)$

or $11 \times 10^{-6} \times (\pi \times 1199.31) \times t = \pi (1200 - 1199.31)$

$t = 52.3° \text{ C}$

1.14 STRAIN ANALYSIS

So far, the effect of an axial force on the length of a bar or rod has been considered. In case of a tensile force, the length increases, and in a compressive force, it decreases. However, this axial increase or decrease takes place at the cost of a change in the lateral dimensions of the bar or rod. If an axial tensile force is applied to a bar, its length is increased and its lateral dimensions i.e. the width and breadth or the diameter are decreased (Fig. 1.32). Therefore, any direct stress produces a strain in its own direction as well as an opposite kind of strain in all directions at right angles to its own direction.

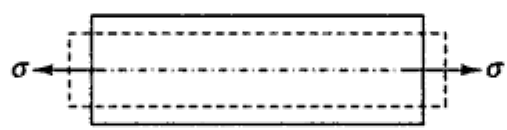


Fig.1.32

Poisson's Ratio

The ratio of the lateral strain to the longitudinal strain of a material, when it is subjected to a longitudinal stress, is known as *Poisson's ratio* and is denoted by ν. It is found that for elastic materials, the lateral strain is proportional to the longitudinal strain i.e. the ratio of the iateral strain to the longitudinal strain is constant. Thus

$$\frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \text{constant} = \nu \tag{1.19}$$

The value of ν lies between 0.25 and 0.34 for most of the metals.

Lateral strain = – $\nu \times$ Longitudinal strain = – $\nu . \sigma/E$

(negative sign indicates that it is opposite to the longitudinal strain)

Two-Dimensional Stress System

Consider a system with two pure normal stresses σ_1 and σ_2 as shown in Fig. 1.33.

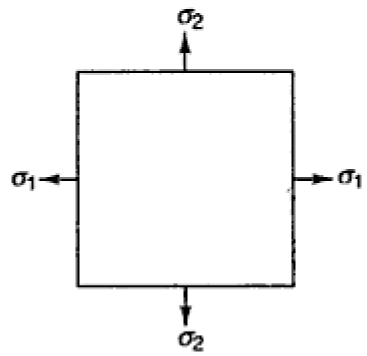


Fig. 1.33

Strain due to σ_1 in its own direction

$= \sigma_1/E$

Strain due to σ_2 in the direction of

$\sigma_1 = -\nu\sigma_2/E$

Thus, net strain in the direction of σ_1

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E \quad (1.20)$$

In a similar way,

Net strain in the direction of σ_2

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_1/E \quad (1.21)$$

Remember that a tensile stress is taken positive whereas a compressive stress negative.

Three-Dimensional Stress System

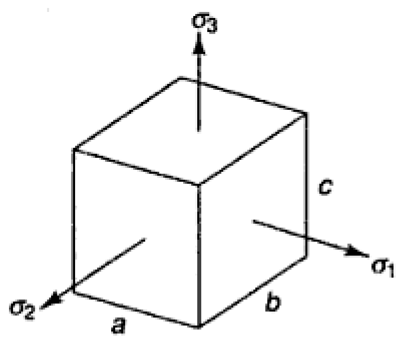


Fig. 1.34

Let there be a system with three pure normal stresses σ_1, σ_2 and σ_3 as shown in Fig. 1.34.

Strain due to σ_1 in its own direction $= \sigma_1/E$

Strain due to σ_2 in the direction of $\sigma_1 = -\nu\sigma_2/E$

Strain due to σ_3 in the direction of $\sigma_1 = -\nu\sigma_3/E$

Thus, the net strain in the direction of σ_1,

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E$$

In a similar way, $\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E$

and $\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E$

Volumetric Strain

Volumetric strain is defined as the ratio of increase in volume of a body to its original volume when it is acted upon by three mutually perpendicular stresses σ_1, σ_2, and σ_3. For a rectangular solid body of sides a, b and c (Fig. 1.34), let ε_1, ε_2 and ε_3 be the corresponding strains.

Initial volume $= a.b.c$

Final volume $= (a + a\varepsilon_1)(b + b\varepsilon_2)(c + c\varepsilon_3) = abc(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{abc(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - abc}{abc}$$

$$= (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - 1$$

$$= 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3 - 1$$

$$\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad (1.22)$$

Thus if the products of very small quantities are neglected, the volumetric strain is the algebraic sum of the three mutually perpendicular strains.

In terms of stresses the volumetric strain can be expressed by substituting the values of ε_1, ε_2 and ε_3 from above.

Volumetric strain,

$$\frac{\Delta V}{V} = (\sigma_1/E - v\sigma_2/E - v\sigma_3/E) + (\sigma_2/E - v\sigma_3/E - v\sigma_1/E) + (\sigma_3/E - v\sigma_1/E - v\sigma_2/E)$$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1-2v)}{E} \tag{1.23}$$

and Change in volume $= V \times \dfrac{(\sigma_1 + \sigma_2 + \sigma_3)(1-2v)}{E}$ (1.24)

If force is unidirectional, i.e. σ_2 and σ_3 are zero,

$$\text{Volumetric strain} = \frac{\sigma(1-2v)}{E} = \varepsilon(1-2v)$$

$$\text{Change in volume} = V \times \frac{\sigma(1-2v)}{E} = V.\varepsilon(1-2v) \tag{1.25}$$

Example 1.22 *What will be the change in the volume of a steel bar of 20-mm diameter and 600-mm length when a tensile stress of 180 MPa is applied to it along its longitudinal axis?*

$$E_s = 205 \text{ GPa}, \; v = 0.3$$

Solution

Volume of the bar, $V = \dfrac{\pi}{4} \times 20^2 \times 600 = 60\,000\,\pi \text{ mm}^3$

Change in volume $= V \times \dfrac{\sigma(1-2v)}{E} = 60\,000\,\pi \times \dfrac{180(1 - 2 \times 0.3)}{205\,000} = 66.2 \text{ mm}^3$

Percentage change in volume $= \dfrac{66.2}{60\,000\,\pi} \times 100 = 0.035$

Example 1.23 *The tangential (hoop) and longitudinal stresses in the plates of a cylindrical boiler of 2.2-m diameter and 3.5-m length are 90 MPa and 45 MPa respectively. Determine the increase in its internal capacity. Neglect compressive stress due to steam on the inner surface.*

$$E = 205 \text{ GPa}; \quad v = 0.3$$

Sozlution

$$\sigma_c = 90 \text{ MPa}; \; \sigma_l = 45 \text{ MPa}$$

$$V = (\pi/4)\, 2.2^2 \times 3.5 = 4.235\,\pi\text{m}^3$$

As compressive stress due to steam on the inner surface is neglected, $\sigma_z = 0$

$$\varepsilon_x = \sigma_c/E - v\sigma_y/E = (1/E)(90 - 0.3 \times 45) = 76.5/E$$

$$\varepsilon_y = \sigma_l/E - v\sigma_x/E = (1/E)(45 - 0.3 \times 90) = 18/E$$

The diameter of a boiler is directly proportional to its circumference. Thus ε_x also is the diametrical strain along any two perpendicular radii.

Volumetric strain, $\varepsilon = \varepsilon_y + \varepsilon_x + \varepsilon_x$

$= 18/E + 76.5/E + 76.5/E$

$= 171/205\,000 = 834.1 \times 10^{-6}$

As strain is a ratio, change in volume can be found directly in m^3.

Change in volume = $\varepsilon \times V = (834.1 \times 10^{-6}) \times 4.235\,\pi = 0.011\,098$ m^3

$= (0.011\,098 \times 1000)\ l = 11.098\ l$

Example 1.24 *A steel bar 35 mm × 35 mm in section and 100 mm long is acted upon by a tensile load of 180 kN along its longitudinal axis and 400 kN and 300 kN along the axes of the lateral surfaces. Determine*

(i) change in the dimensions of the bar (ii) change in volume

(iii) longitudinal axial load acting alone to produce the same longitudinal strain as in (i).

$E = 205$ GPa; $\nu = 0.3$

Solution Let σ_1, σ_2 and σ_3 be the stresses along longitudinal and two transverse axes respectively.

(i)
$$\sigma_1 = \frac{180\,000}{35 \times 35} = 146.9 \text{ MPa}$$

$$\sigma_2 = \frac{400\,000}{100 \times 35} = 114.3 \text{ MPa}$$

$$\sigma_3 = \frac{300\,000}{100 \times 35} = 85.7 \text{ MPa}$$

In longitudinal direction,

$$\Delta L = \frac{L}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$= \frac{100}{205\,000}(146.9 - 0.3 \times 114.3 - 0.3 \times 85.7) = 0.042\,39 \text{ mm}$$

In the direction of 400 kN load,

$$\Delta L = \frac{L}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3)$$

$$= \frac{35}{205\,000}(114.3 - 0.3 \times 146.9 - 0.3 \times 85.7) = 0.0076 \text{ mm}$$

In the direction of 300 kN load,

$$\Delta L = \frac{L}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

$$= \frac{35}{205\,000}(85.7 - 0.3 \times 146.9 - 0.3 \times 114.3) = 0.001\,25 \text{ mm}$$

(ii) Change in volume = $(100 + 0.042\,39)(35 + 0.0076)(35 + 0.001\,25) - 100 \times 35 \times 35$

$= 82.92$ mm^3

or Change in volume = $V \times \dfrac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E}$

$$= (100 \times 35 \times 35)\,\frac{(146.9 + 114.3 + 85.7)(1 - 2 \times 0.3)}{205\,000}$$

$= 82.92$ mm^3

(iii) Let σ be the longitudinal stress to have the same strain,

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.042\ 39}{100} = \frac{\sigma}{205\ 000}, \ \sigma = 86.9 \text{ MPa}$$

Longitudinal load = $\sigma \times A = 86.9 \times 35 \times 35 = 106\ 452$ N or 106.452 kN

Example 1.25 *A square steel bar of dimensions 50 mm × 50 mm × 150 mm is subjected to an axial load of 250 kN. Determine the decrease in length of the bar if*

(i) the lateral strain is fully prevented by applying external uniform pressure on the rectangular surfaces.

(ii) only one-third of the lateral strain is prevented by the external pressure.

Solution

(i) $\sigma_1 = 250\ 000/(50 \times 50) = 100$ MPa

Let the compressive stresses applied on the similar lateral sides be σ_2 (= σ_3) to prevent the lateral strain (Fig. 1.35). Then

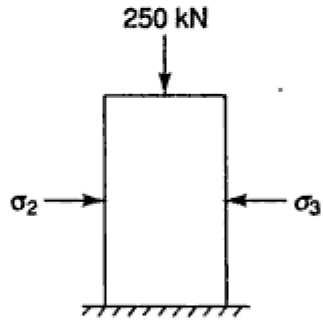


Fig. 1.35

$$\frac{1}{E}(\sigma_2 - \nu\sigma_3 - \nu\sigma_1) = 0$$

or $(\sigma_2 - 0.3 \times \sigma_2 - 0.3 \times 100) = 0$ ....($\sigma_2 = \sigma_3$)

or $0.7\ \sigma_2 = 30$

$\sigma_2 = 42.857$ MPa

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = 0$$

$$= \frac{150}{205\ 000}(100 - 0.3 \times 2 \times 42.857) = 0 \quad ....(\sigma_2 = \sigma_3)$$

$= 0.054\ 36$ mm

(ii) In the absence of compressive stresses on the sides to prevent the lateral strain,

The lateral strain = $\nu\sigma_1/E$ (tensile)

Now, one-third of this is to be prevented i.e. $\nu\sigma_1/3E$ and leaving $2\nu\sigma_1/3E$ as such. Let the compressive stresses applied on the sides be σ_2.

Then

$$\frac{1}{E}(\sigma_2 - \nu\sigma_3 - \nu\sigma_1) = -\frac{2\nu\sigma_1}{3E}$$

The two strains are of opposite directions.

or $(\sigma_2 - 0.3 \times \sigma_2 - 0.3 \times 100) = -2 \times 0.3 \times 100/3$ ($\sigma_2 = \sigma_3$)

or $0.7\ \sigma_2 = 30 - 20$

$\sigma_2 = 14.286$ MPa

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) = 0$$

$$= \frac{150}{205\ 000}(100 - 0.3 \times 2 \times 14.286) = 0 \quad (\sigma_2 = \sigma_3)$$

$= 0.0669$ mm

1.15 TENSILE TEST DIAGRAM

The behaviour of a ductile material, such as steel, subjected to an increased tensile load is studied by testing a specimen in a tensile testing machine. The plot between strain and the corresponding stress is represented graphically by a tensile test diagram. Figure 1.36 shows a stress vs. strain diagram for steel in which the stress is calculated on the basis of original area of a steel bar. Most of the other engineering materials show a similar pattern to a varying degree. The following are the salient features of the diagram:

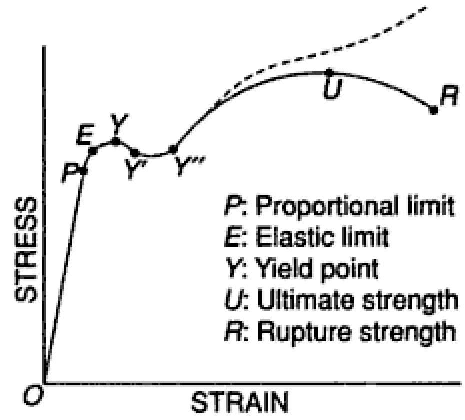


Fig. 1.36

- When the load is increased gradually, the strain is proportional to load or stress upto a certain value. Line *OP* indicates this range and is known as the *line of proportionality*. Hooke's law is applicable in this range. The stress at the end point *P* is known as the *proportional limit.*
- If the load is increased beyond the limit of proportionality, the elongation is found to be more rapid, though the material may still be in the elastic state, i.e. on removing the load, the strain vanishes. The point *E* depicts the elastic limit. Hooke's law cannot be applied in this range as the strain is not proportional to stress. Usually, this point is very near to *P* and many times the difference between *P* and *E* is ignored and therefore elastic limit is taken as the limit of proportionality.
- When the load is further increased, plastic deformation occurs i.e. on removing the load, the strain is not fully recoverable. At point *Y*, metal shows an appreciable strain even without further increasing the load. Actually, the curve drops slightly at this point to Y' and the yielding goes upto point Y''. The points Y' and Y'' are known as the *upper* and *lower yield points* respectively. The stress-strain curve between *Y* and Y'' is not steady.
- After the yield point, further straining is possible only by increasing the load. The stress strain curve rises upto point *U*, the strain in the region *Y* to *U* is about 100 times that from *O* to Y'. The stress value at *U* is known as the *ultimate stress* and is mostly plastic which is not recoverable.
- If the bar is stressed further, it begins to form a *neck* or a local reduction in cross-section occurs. After this, somewhat lower loads are sufficient to keep the specimen elongating further. Ultimately, the specimen fractures at point *R*.
- If the load is divided by the original area of the cross-section, the stress is known as the *nominal stress*. This is lesser at the rupture load than at the maximum load. However, the stress obtained by dividing with the reduced area of cross-section is known as the *actual* or *true* stress and is greater at the maximum load. It is shown in the figure by the dotted line.

  For more information refer to section 18.3.

1.16 FACTOR OF SAFETY

A machine component must be designed so that the load carried by it under normal conditions of utilisation is considerably below its ultimate load. This smaller load is referred as the *allowable load* or the *design load* or the *working load*. Usually, the allowable load is only a fraction of the ultimate load or the load carrying capacity of a component. This is done to ensure safe working of the component against uncertainties of various factors during the operation of a machine, e.g., homogeneousness of the material, number of loadings during the life of component, type of loading (static or sudden), method of analysis used, natural causes, etc. Thus a large portion of the load carrying capacity of the component is kept as reserve for safe performance of the component. The ratio of the ultimate load to the allowable or working load is known as factor of safety. Thus

$$\text{Factor of safety} = \frac{\text{ultimate load}}{\text{allowable load}}$$

As the stress is the load per unit area, factor of safety is also defined as,

$$\text{Factor of safety} = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

- Refer section 18.3 also.

1.17 ELASTIC CONSTANTS

The factors to determine the deformations produced by a stress system acting on a material within elastic limits are constant and termed as *elastic constants*. Two elastic constants *Modulus of Elasticity* and *Modulus of Rigidity* have already been defined in section 1.5. A third elastic constant is being defined in this section.

If three mutually perpendicular stresses of equal intensity are applied to a body of initial volume V as shown in Fig. 1.37, then the ratio of the direct stress to the volumetric strain is known as the *bulk modulus* (K) of the body.

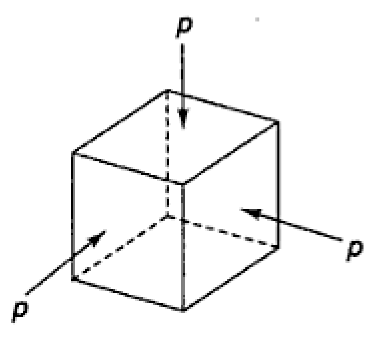


Fig. 1.37

Usually, bulk modulus is applicable mainly to fluid problems with pressure intensity p in all directions and thus

$$K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{-p}{\varepsilon_v} \tag{1.26}$$

$$\text{Volumetric strain, } \varepsilon_v = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E} \qquad \text{(Eq. 1.23)}$$

For three perpendicular stresses of equal intensity p (compressive),

$$\varepsilon_v = \frac{3(1 - 2\nu)(-p)}{E}$$

Therefore, $K = \dfrac{-p}{-3(1-2\nu)p/E}$

or $E = 3K(1-2\nu)$ (1.27)

1.18 RELATION BETWEEN ELASTIC CONSTANTS

Consider a square element *ABCD* under the action of a simple shear stress τ(Fig.1.38*a*). The resultant distortion of the element is shown in Fig. 1.38*b*. The total change in the corner angles is $\pm\varphi$. However, for convenience sake, side *AB* may be considered to be fixed as shown in Fig. 1.38*c*. As angle φ is extremely small, *CC′* and *DD′* can be assumed to be arcs. Let *CE* be a perpendicular on the diagonal *AC′*.

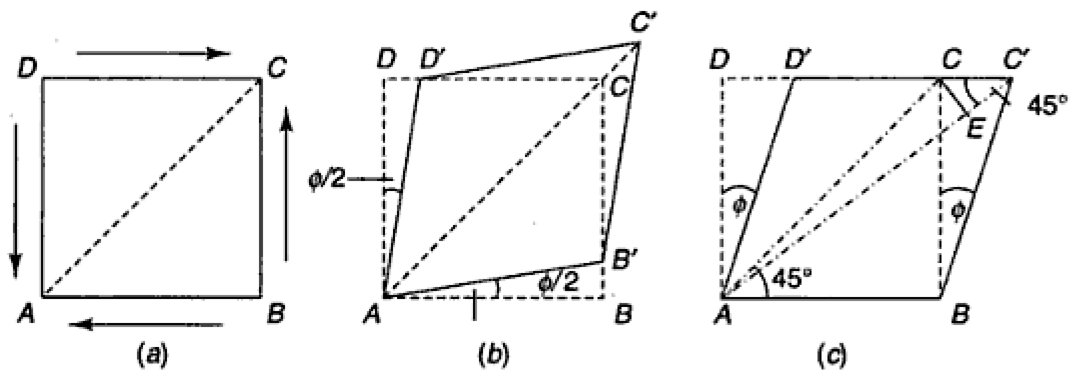


Fig. 1.38

Linear strain of the diagonal *AC* can approximately be taken as

$$\varepsilon = \frac{AC' - AC}{AC} = \frac{EC'}{AC} = \frac{CC'\cos 45°}{AB/\cos 45°} = \frac{\varphi.BC\cos^2 45°}{BC} = \frac{\varphi}{2}$$

($CC' = \varphi.BC$ and $AB = BC$)

But Modulus of rigidity, $G = \tau/\varphi$ or $\varphi = \tau/G$ (Eq.1.5)

$\therefore$ $\varepsilon = \dfrac{\tau}{2G}$ (i)

It will be shown in section 2.2 that in a state of simple shear on two perpendicular planes, the planes at 45° are subjected to a tensile stress (magnitude equal to that of he shear stress) while the planes at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes. Thus planes *AC* and *BD* are subjected to tensile and compressive stresses respectively each equal to τ in magnitude as shown in Fig. 1.39.

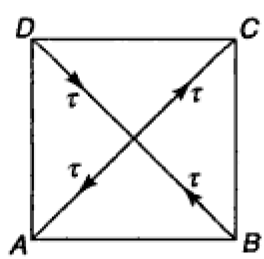


Fig.1.39

Hence linear strain of diagonal *AC* is

$$\varepsilon = \frac{\tau}{E} - \left(-\frac{\nu\tau}{E}\right) = \frac{\tau}{E}(1+\nu)$$ (ii)

From (i) and (ii),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

or $E = 2G(1 + \nu)$

As $E = 3K(1 - 2\nu)$ ...(Eq. 1.27)

$\therefore$ $E = 2G(1 + \nu) = 3K(1 - 2\nu)$ (1.28)

This equation relates the elastic constants.

Also from above, $1 + \nu = \frac{E}{2G}$, $\therefore$ $2 + 2\nu = \frac{E}{G}$ (i)

and $1 - 2\nu = \frac{E}{3K}$ (ii)

Adding (i) and (ii), $3 = E\left(\frac{1}{G} + \frac{1}{3K}\right) = \frac{E}{3KG}(3K + G)$

or $E = \frac{9KG}{3K + G}$ (1.29)

Example 1.26 *A bar, 24 mm in diameter and 400 mm in length, is acted upon by an axial load of 38 kN. The elongation of the bar and the change in diameter are measured as 0.165 mm and 0.0031 mm respectively. Determine*

(i) Poisson's ratio, and (ii) the values of the three moduli

Solution

$$A = (\pi/4)\, 24^2 = 144\, \pi \text{ mm}^2$$

$$\sigma = 38\,000/144\,\pi = 84 \text{ MPa}$$

Lateral strain = ν .Linear strain

$$\frac{\delta d}{d} = \nu\frac{\delta L}{L} \text{ or } \frac{0.0031}{24} = \nu\frac{0.165}{400} \text{ or } \nu = 0.313$$

$$E = \frac{\sigma}{\varepsilon} = \frac{84}{0.165/400} = 203\,636 \text{ MPa}$$

Also, $E = 2\,G\,(1 + \nu) = 3K\,(1 - 2\nu)$

$\therefore$ $$G = \frac{E}{2(1+\nu)} = \frac{203\,636}{2(1+0.313)} = 77\,546 \text{ MPa}$$

and $$K = \frac{E}{3(1-2\nu)} = \frac{203\,636}{3(1-2\times 0.313)} = 181\,494 \text{ MPa}$$

Example 1.27 *A bar, 12 mm in diameter, is acted upon by an axial load of 20 kN. The change in diameter is measured as 0.003 mm. Determine*

(i) Poisson's ratio and (ii) the modulus of elasticity and the bulk modulus.

The value of the modulus of rigidity is 80 GPa.

Solution

$$A = (\pi/4)\, 12^2 = 36\, \pi \text{ mm}^2$$

$$\sigma = 20\,000/36\,\pi = 176.84 \text{ MPa}$$

Lateral strain = ν . Linear strain

$$\frac{\delta d}{d} = \nu\varepsilon \quad \text{or} \quad \frac{0.003}{12} = \nu\varepsilon \quad \text{or} \quad \varepsilon = 0.000\,25/\nu \qquad \text{(i)}$$

Now, $E = 2G(1 + \nu) = 2 \times 80\,000\,(1 + \nu) = 160\,000 + 160\,000\,\nu$

Also, $$E = \frac{\sigma}{\varepsilon} = \frac{176.84}{0.000\,25/\nu} = 707\,360\,\nu \qquad \text{[using (i)]}$$

$\therefore$ $707\,360\,\nu = 160\,000 + 160\,000\,\nu$

$\nu = 0.2923$

Thus $E = 707\,360 \times 0.2923 = 206\,771$ MPa

and $$K = \frac{E}{3(1-2\nu)} = \frac{206\,771}{3(1-2\times 0.2923)} = 165\,921 \text{ MPa}$$

Summary

- The resisting force per unit area of cross-section of a body is known as *intensity of stress* or simply *stress*.
- *Shear stress* exists on two parts of a body when two equal and opposite parallel forces, not in the same line, act and one part tends to slide over or shear from the other across any section.
- The elongation per unit length of a body is known as *strain*. It is dimensionless.
- *Shear strain* is the change in the right angle of a rectangular element measured in radians and is dimensionless.
- The ratio of stress to strain is constant within elastic limits and is known as *Young's Modulus* or the *Modulus of Elasticity*. It is denoted by E and $E = \sigma/\varepsilon$.
- *Modulus of rigidity* or *shear modulus* denoted by G is the ratio of shear stress to shear strain i.e. $G = \tau/\varphi$.
- The materials in which strain is proportional to stress are said to obey the *Hook's law*.
- Elongation of a bar of length L is given by, $\Delta = PL/AE$
- The principle of superposition states that if a body is acted upon by a number of loads, then the net effect on the body is the sum of the effects caused by each load acting independently.
- Increase in length due to temperature rise = $L.\alpha.t$
- Temperature stress is given by $\sigma = \alpha\, t\, E$
- For elastic materials, the ratio of lateral strain to longitudinal strain is constant and is known as *Poisson's ratio* (ν).
- Volumetric strain is the ratio of increase in volume of a body to its original volume when it is acted upon by three mutually perpendicular stresses.
- Volumetric strain = $\dfrac{(\sigma_1 + \sigma_2 + \sigma_3)(1-2\nu)}{E} \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

- Factor of safety = $\dfrac{\text{ultimate load}}{\text{allowable load}}$ or $\dfrac{\text{ultimate stress}}{\text{allowable stress}}$
- Relation between various elasticity constants of a material is
  $E = 2G(1+ \nu) = 3K(1 - 2\nu)$

Review Questions

1. What do you mean by tensile, compressive and shear forces? Give examples.
2. What is stress? In what way does shear stress differ from direct stress? Explain.
3. What is complimentary shear stress? What is its significance?
4. Explain the terms: strain, shear strain, Young's modulus and modulus of rigidity.
5. Define the principle of superposition. What is its utility?
6. Deduce expressions to determine the elongation of
   (i) a bar of tapering section, and
   (ii) a trapezoidal section of uniform thickness.
7. Find an expression for elongation of a bar of rectangular section and a conical section due to self-weight.
8. What is meant by a column of uniform strength? How is its area of cross-section along its length related to that at the top?
9. What are compound bars? What are equilibrium and compatibility equations?
10. What do you mean by temperature stresses? Explain.
11. Define the term Poisson's ratio. Write the expressions for strains in the three principal directions.
12. What is volumetric strain? Show that it is the algebraic sum of three mutually perpendicular strains.
13. Plot a tensile test diagram for steel. Explain its salient features.
14. Define the term *factor of safety* and its importance.
15. Define bulk modulus. Deduce the relation $E = 3K(1- 2\nu)$.
16. Derive a relation between Young's modulus, modulus of rigidity and Poisson's ratio.
17. A rod as shown in Fig. 1.40 is subjected to a tensile force of 125 kN. Determine the elongation of the rod. $E = 205$ GPa. (0.7 mm)

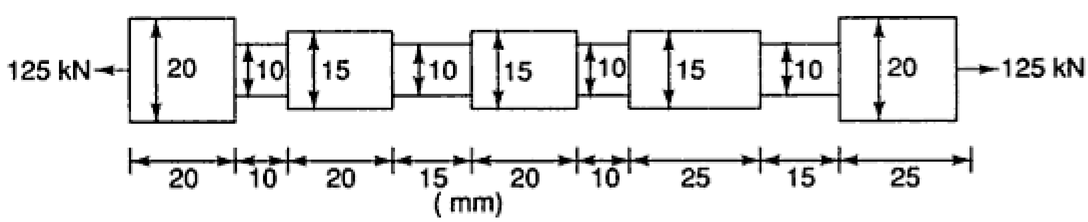


Fig. 1.40

18. The loading on a steel bar of 30-mm diameter is as shown in Fig. 1.41. Find the elongation of the bar. E = 205 GPa. (0.224 mm)

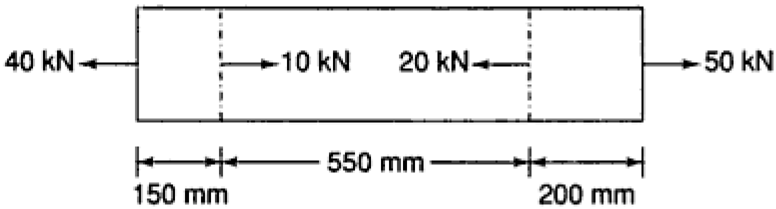


Fig. 1.41

19. Determine the total compression of the bar of Fig. 1.42. Take E_s = 210 GPa, E_b = 105 GPa and E_c = 100 GPa. (0.15 mm)

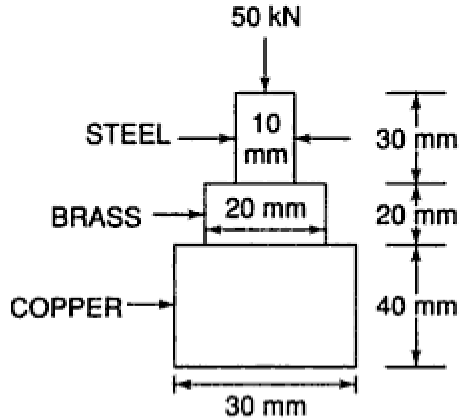


Fig. 1.42

20. A flange coupling is used to join two shafts which are to transmit 300 kW at 800 rpm. 8 bolts are used at a pitch diameter of 120 mm. Assuming a mean shear stress of 80 MPa, determine the diameter of the bolts. (9 mm)
21. A 24-mm steel rod passes centrally through a copper tube of 36-mm external diameter, 30-mm internal diameter and 80-mm length. The tube is closed at each end by rigid washers and nuts screwed to the rod. The nuts are tightened till the compressive force in the copper tube is 25 kN. Determine the stresses in the rod and the tube.
(55.24 MPa tensile; 80.38 MPa compressive)
22. A load of 800 kN is applied to a reinforced concrete column of 560-mm diameter which has four steel rods of 36-mm diameter embedded in it. Determine the stress in the concrete and the steel. Take E for steel = 210 GPa and E for concrete = 15 GPa.
Also find the adhesive force between the concrete and the steel.
[2.67 MPa (concrete); 37.43 MPa (steel); 139.3 kN]

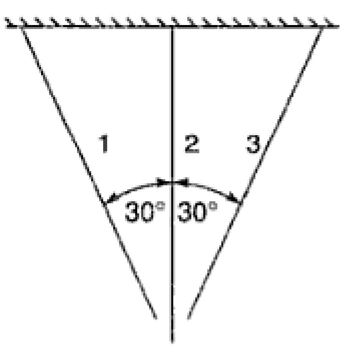


Fig. 1.43

23. In a framed structure shown in Fig. 1.43, all the three rods are of the same material and with the same area of cross section of 120 mm^2. The central rod is 240 mm long but is longer than its requirement by 2 mm. Determine the forces in the rods if their lower ends are welded together. E = 205 GPa.
(115.87 kN in the rod 2 and 66.9 kN in rods 1 and 3)

24. In the framed structure of Fig. 1.43, the outer rods are of steel and of 260-mm^2 area of cross section whereas the central rod is of brass and of 420-mm^2 area of cross-section. The length of the central rod is 1200 mm. Initially, all the rods are of required length. However, while assembling, the central rod is heated through 40°C. Determine the stresses developed in the rods. E for steel = 205 GPa, E for brass = 85 GPa and α for brass = 19×10^{-6}/°C.

(17.9 kN in 2; 10.34 kN in 1 and 3)

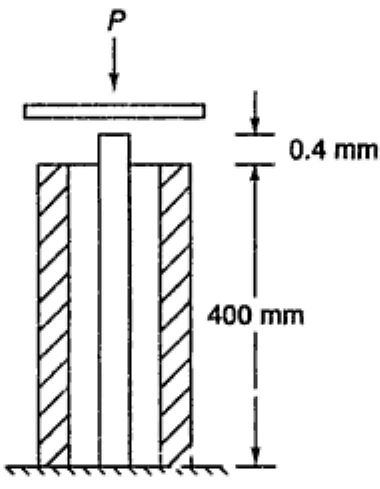


Fig. 1.44

25. A steel sleeve of 24-mm internal diameter and 36-mm external diameter encloses an aluminum rod of 22-mm diameter. The length of the rod is 0.4 mm longer than that of the sleeve which is 400 mm long as shown in Fig. 1.44. Determine

(i) the compressive load up to which only the rod is stressed

(ii) the maximum load on the assembly, if the permissible stresses in aluminum and steel are 130 MPa and 175 MPa respectively

(iii) the deformation of the assembly under maximum load

E_a = 75 GPa and E_s = 205 GPa

(17.42 kN; 138.71 kN; 0.3416 mm)

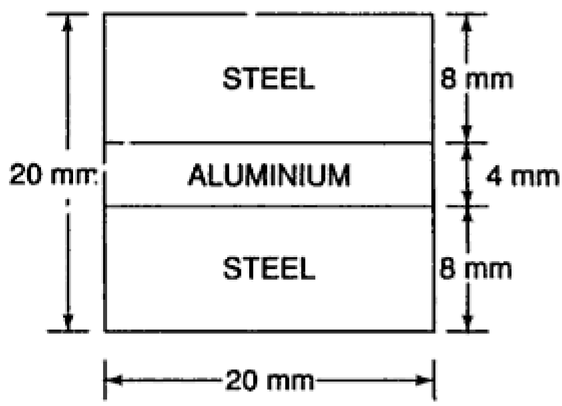


Fig. 1.45

26. A composite bar of 20 mm × 20 mm cross section is made up of three flat bars as shown in Fig. 1.45. All the three bars are rigidly connected at the ends when the temperature is 20°C. Determine

(i) the stresses developed in each bar when the temperature of the composite bar is raised to 60°C

(ii) the final stresses in each bar when a load of 17.6 kN is applied to the composite bar

E_a = 80 GPa, $\alpha_a = 11 \times 10^{-6}$/°C

E_s = 200 GPa, $\alpha_s = 22 \times 10^{-6}$/°C

(σ_s = 8 MPa; σ_a = 32 MPa; σ_s = 42 MPa; σ_a = 52 MPa)

27. A load of 120 kN is applied to a bar of 20-mm diameter. The bar which is 400-mm long is elongated by 0.7 mm. Determine the modulus of elasticity of the bar material. If Poisson's ratio is 0.3, find the change in diameter.

(218 GPa; 0.0105 mm)

28. A metallic prismatic specimen is subjected to an axial stress of σ_x and on one pair of sides no constraint is exerted whereas on the other, the lateral strain is restricted to one half of the strain which would be there in the absence of any constraint. Show that the modified modulus of elasticity will be $[2E/(2-\nu^2)]$.

29. A prismic bar is stretched in such a way that all the lateral strain is prevented. Find the value of the modified modulus of elasticity. What will be the modefied Poisson's ratio? $[E(1-\nu)/(1+\nu)(1-2\nu)$; zero]

30. A steel bar of 10 mm diameter is subjected to an axial load of 12 kN. If the change in diameter is found to be 0.0022 mm, determine Poisson's ratio, the modulus of elasticity and the bulk modulus. Take $G = 78$ GPa.

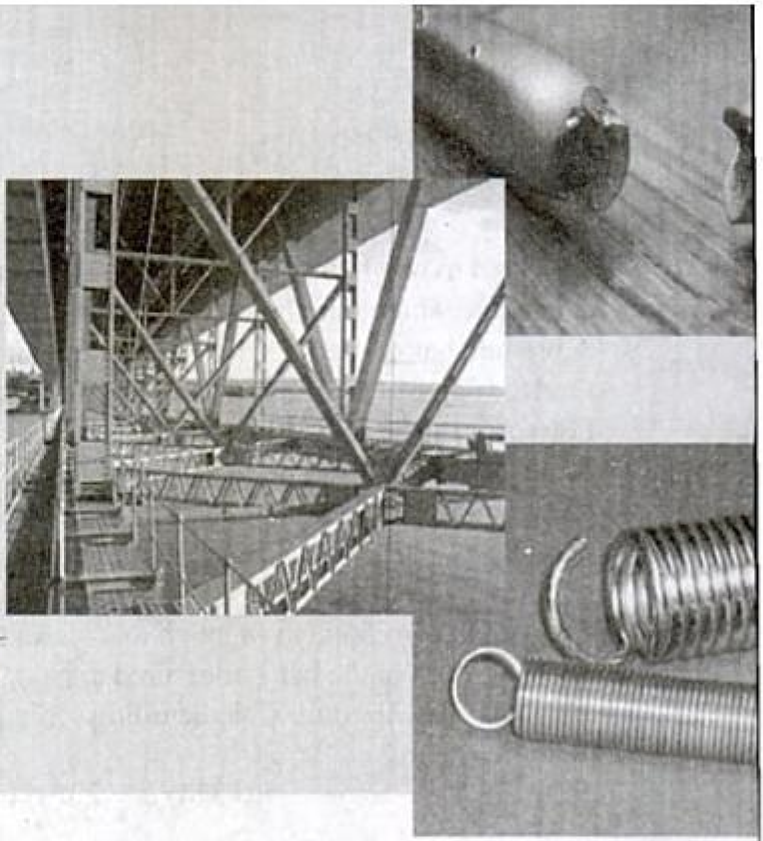
(0.29; 201.4 GPa, 159.8 GPa)

31. An axial load of 56 kN is applied to a bar of 36-mm diameter and 1-m length. The extension of the bar is measured to be 0.265 mm whereas the reduction in diameter is 0.003 mm. Calculate Poisson's ratio and the values of the three moduli.

(0.314, $E = 207.6$ GPa, $G = 79$ GPa , $K = 186$ GPa)

2

COMPOUND STRESS AND STRAIN



2.1 INTRODUCTION

In the first chapter, direct and shear forces were assumed to act independently. Also, the stresses were determined on planes in the normal or tangential directions. However, in most of the cases, direct and shear forces act simultaneously on a body and the maximum value of the resultant stress may act in some other direction than of the load application. It is, therefore, necessary to find out stresses on planes other than those of load application.

In practical problems, the stress varies from point to point in a loaded member. Therefore, the equilibrium of an element at a point is to be considered by taking the element of infinitesimal dimensions so that the stresses approach the conditions at the point. Such an infinitesimal element may be considered of any convenient shape and the stresses may be assumed uniformly distributed over the surface of the element. In a two-dimensional analysis, the thickness of the element does not affect the results and for convenience sake may be taken unity.

2.2 STRESS ANALYSIS

While analysing a stress system, the general conventions have been taken as

- a tensile stress is positive and compressive stress negative.
- a pair of shear stresses on parallel planes forming a clockwise couple is positive and a pair with counter-clockwise couple negative.
- clockwise angle is taken as positive and counter-clockwise negative.

The following cases are being considered:

(i) Direct stress condition (ii) Bi-axial stress condition
(iii) Pure shear stress condition (iv) Bi-axial and shear stresses condition

(i) Direct Stress Condition

Let a bar be acted upon by an external force P and thus a tensile stress acts along its length (Fig. 2.1*a*). The stress on any transverse section such as BC will have pure normal stress acting on it. The stress acting on an arbitrary plane AC inclined at an angle θ with the vertical plane BC will have two components:

- normal component known as *direct stress component* and
- tangential component known as *shear stress component*.

These stress components can be determined from the consideration of force balance.

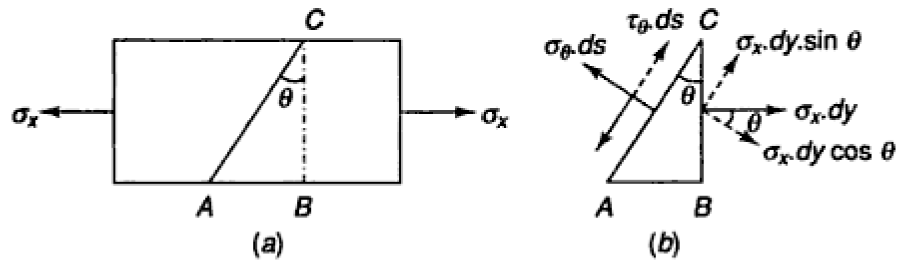


Fig. 2.1

If the bar is imagined to be cut through the section AC, each portion is in equilibrium under the action of external load P and the stresses on plane AC. For convenience, a triangular prismatic element ABC containing the plane AC can be taken for the force analysis.

Figure 2.1*b* shows the forces acting on the triangular element.

Let

dy = the length of the side BC
ds = the length of the side AC
σ_x = normal stress acting on the plane BC
σ_θ = normal stress acting on the plane AC
τ_θ = tangential or shear stress acting on the plane AC

Assume a unit thickness of the prism and equate the forces along normal and tangential directions to the plane AC of the prism for its equilibrium, i.e.

$$\sigma_\theta.ds - \sigma_x.dy.\cos\theta = 0$$

or
$$\sigma_\theta = \frac{\sigma_x dy\cos\theta}{ds} = \frac{\sigma_x dy\cos\theta}{dy/\cos\theta} = \sigma_x\cos^2\theta \tag{2.1}$$

and
$$\tau_\theta.ds + \sigma_x.dy.\sin\theta = 0 \quad \text{(assuming } \tau_\theta \text{ clockwise as positive)}$$

$$\tau_\theta = -\frac{\sigma_x dy\sin\theta}{ds} = -\frac{\sigma_x dy\sin\theta}{dy/\cos\theta}$$
$$= -\sigma_x\sin\theta\cos\theta = -\frac{1}{2}\sigma_x\sin 2\theta \tag{2.2}$$

The negative sign shows that τ_θ is counter-clockwise and not clockwise on the inclined plane.

- When $\theta = 0°$, $\sigma_\theta = \sigma_x$ and $\tau_\theta = 0$
- When $\theta = 45°$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = -\sigma_x/2$ (maximum, counter-clockwise)

- When $\theta = 90°$, $\sigma_\theta = 0$ and $\tau_\theta = 0$
- When $\theta = 135°$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = \sigma_x/2$ (maximum, clockwise)

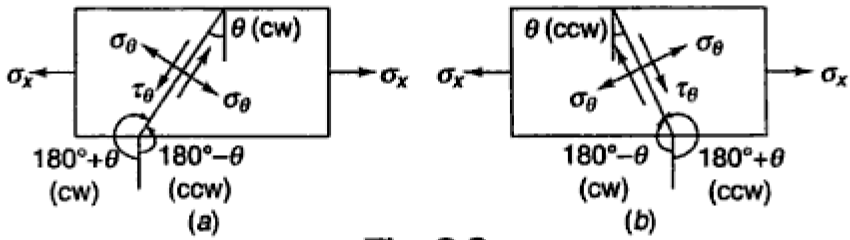


Fig. 2.2

Figures 2.2 (*a*) and (*b*) show the planes inclined at different angles to the vertical alongwith the stresses acting on them. It can be noted from these figures alongwith the above observations that

- a plane at angle θ with the vertical also is the plane with angle $(180° + \theta)$. Thus a plane at angle 45° clockwise with vertical can also be mentioned as the plane at 225° clockwise or 135° counter-clockwise. Similarly, a plane at angle – 45° with the vertical would also mean a plane at angle 45° counter-clockwise or angle 225° counter-clockwise or angle 135° clockwise.
- the normal stress on the inclined plane decreases with the increase in angle θ, from maximum on the vertical plane to zero on the horizontal plane.
- the shear stress is negative (counter-clockwise) between 0° and 90° and positive (clockwise) between 0° and – 90°. Remember that plane at 135° to the vertical also means a plane at – 45° as described above.
- the maximum shear stress is equal to one half the applied stress.

The resultant stress on the plane *AC*,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sigma_x\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}$$

$$= \sigma_x \cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \sigma_x \cos\theta \tag{2.3}$$

Inclination with the normal stress,

$$\tan\varphi = \frac{\sigma_x \sin\theta\cos\theta}{\sigma_x\cos^2\theta} = \tan\theta$$

or $$\varphi = \theta \tag{2.4}$$

That is, it is always in the direction of the applied stress.

(ii) Bi-axial Stress Condition

Let an element of a body be acted upon by two tensile stresses acting on two perpendicular planes of the body as shown in Fig. 2.3. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively.

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

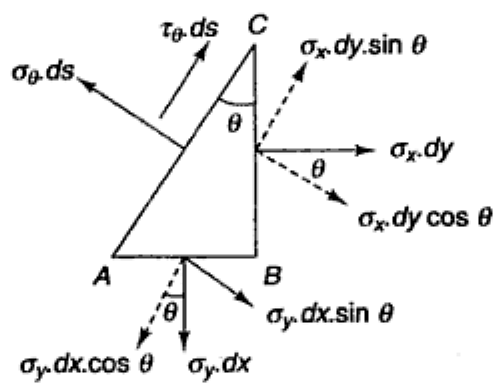


Fig. 2.3

$$\sigma_\theta .ds - \sigma_x.dy.\cos\theta - \sigma_y.dx.\sin\theta = 0$$

or
$$\sigma_\theta = \frac{\sigma_x dy\cos\theta}{ds} + \frac{\sigma_y dx\sin\theta}{ds} = \frac{\sigma_x dy\cos\theta}{dy/\cos\theta} + \frac{\sigma_y dx\sin\theta}{dx/\sin\theta}$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \qquad (2.5)$$

The expression may be put in the following form,

$$\sigma_\theta = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \qquad (2.6)$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta .ds + \sigma_x .dy.\sin\theta - \sigma_y .dx.\cos\theta = 0$$

or
$$\tau_\theta = -\frac{\sigma_x dy\sin\theta}{ds} + \frac{\sigma_y dx\cos\theta}{ds} = -\frac{\sigma_x dy\sin}{dy/\cos\theta} + \frac{\sigma_y dx\cos\theta}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta = -(\sigma_x - \sigma_y)\sin\theta\cos\theta$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \qquad (2.7)$$

Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^{\ 2} + \tau_\theta^{\ 2}}$$

$$= \left[\left\{\frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta\right\}^2 + \left\{-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2\cos^2 2\theta + 2.\frac{1}{2}.\frac{1}{2}(\sigma_x + \sigma_y)(\sigma_x - \sigma_y)\cos 2\theta + \frac{1}{4}(\sigma_x - \sigma_y)^2\sin^2 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}\left\{(\sigma_x - \sigma_y)^2(\cos^2 2\theta + \sin^2 2\theta)\right\} + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x + \sigma_y)^2 + \frac{1}{4}(\sigma_x - \sigma_y)^2 + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y + \sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y) + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}(\sigma_x^2 + \sigma_y^2) + \frac{1}{2}(\sigma_x^2 - \sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2(1+\cos 2\theta)+\frac{1}{2}\sigma_y^2(1-\cos 2\theta)\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2.2\cos^2\theta+\frac{1}{2}\sigma_y^2.2\sin^2\theta\right]^{1/2}$$

$$= \sqrt{\sigma_x^2\cos^2\theta+\sigma_y^2\sin^2\theta} \qquad (2.8)$$

and the angle of inclination of the resultant with σ_θ,

$$\tan\varphi = \frac{\tau_\theta}{\sigma_\theta} = \frac{-(\sigma_x-\sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta+\sigma_y\sin^2\theta} = \frac{\sigma_y-\sigma_x}{\sigma_x\cot\theta+\sigma_y\tan\theta} \qquad (2.9)$$

For greatest obliquity or inclination of the resultant with the normal stress,

$$\frac{d(\tan\varphi)}{d\theta} = 0$$

or $\quad -\sigma_x\text{cosec}^2\theta+\sigma_y\sec^2\theta = 0$ or $\sigma_x\text{cosec}^2\theta = \sigma_y\sec^2\theta$

$$\tan^2\theta = \frac{\sigma_x}{\sigma_y} \quad \text{or} \quad \tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}} \qquad (2.10)$$

$$\therefore \qquad \tan\varphi_{\max} = \frac{\sigma_y-\sigma_x}{\sigma_x\sqrt{\sigma_y/\sigma_x}+\sigma_y\sqrt{\sigma_x/\sigma_y}} = \frac{\sigma_y-\sigma_x}{2\sqrt{\sigma_x\sigma_y}} \qquad (2.10a)$$

The angle of inclination of the resultant with σ_x,

$$\tan\alpha = \frac{\sigma_y.dx}{\sigma_x.dy} = \frac{\sigma_y.dy.\tan\theta}{\sigma_x.dy} = \frac{\sigma_y}{\sigma_x}\tan\theta \qquad (2.11)$$

The above results show that

- the normal stress on the inclined plane varies between the values of σ_x and σ_y as the angle θ is increased from 0° to 90°. For equal values of the two axial stresses ($\sigma_x = \sigma_y$), σ_θ is always equal to σ_x or σ_y.
- the shear stress is zero on planes with angles 0° and 90°, i.e. on horizontal and vertical planes. It has maximum value numerically equal to one half the difference between given normal stresses which occurs on planes at ± 45° to the given planes.

$$\tau_{\max} = \pm\frac{1}{2}(\sigma_x-\sigma_y) \qquad (2.12)$$

  and the normal stress across the same plane,

$$\sigma_{45^\circ} = \frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}(\sigma_x-\sigma_y)\cos 90^\circ = \frac{1}{2}(\sigma_x+\sigma_y) \qquad (2.13)$$

- shear stress in a body subjected to equal perpendicular stresses is zero (Refer Eq. 2.7).
- if any of the given stresses is compressive, the stress can be replaced by a negative sign in the above derived expressions i.e. σ_x with $-\sigma_x$ and σ_y with $-\sigma_y$.

- if σ_y is compressive, the maximum value of shear stress across a plane at 45° plane is

$$\tau_{max} = \frac{1}{2}[(\sigma_x - (-\sigma_y)] = \frac{1}{2}(\sigma_x + \sigma_y)$$

and if σ_x is numerically equal to σ_y,

$$\tau_{max} = \sigma_x = \sigma_y \tag{2.14}$$

(iii) Pure Shear Stress Condition

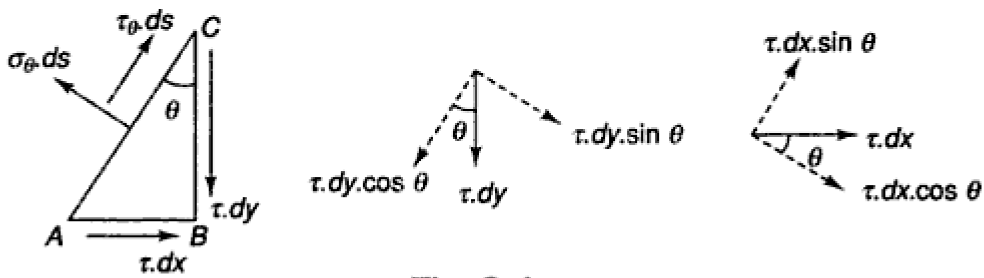


Fig. 2.4

Let an element of a body be acted upon by shear stresses on its two perpendicular faces as shown in Fig. 2.4. Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta.ds - \tau.dx.\cos\theta - \tau.dy.\sin\theta = 0$$

or
$$\sigma_\theta = \frac{\tau\, dx \cos\theta}{ds} + \frac{\tau\, dy \sin\theta}{ds} = \frac{\tau\, dx\cos\theta}{dx/\sin\theta} + \frac{\tau\, dy \sin\theta}{dy/\cos\theta}$$

$$= \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta = \tau.\sin 2\theta \tag{2.15}$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta .ds - \tau.dy.\cos\theta + \tau.dx.\sin\theta = 0$$

or
$$\tau_\theta = \frac{\tau\, dy\cos\theta}{ds} - \frac{\tau\, dx \sin\theta}{ds} = \frac{\tau\, dy\cos\theta}{dy/\cos\theta} - \frac{\tau\, dx\sin\theta}{dx/\sin\theta}$$

$$= \tau\cos^2\theta - \tau\sin^2\theta$$

$$= \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right] = \tau\cos 2\theta \tag{2.16}$$

which shows that it is up the plane for $\theta < 45°$ and down the plane for $\theta > 45°$.

The resultant stress on the plane AC,

$$\sigma_r = \sqrt{\sigma_\theta{}^2 + \tau_\theta{}^2} = \tau\sqrt{(\sin 2\theta)^2 + (\cos 2\theta)^2} = \tau \tag{2.17}$$

Inclination with the direction of shear stress planes,

$$\tan\varphi = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

or
$$\varphi = 2\theta \tag{2.18}$$

The above equations show that

- the normal stress is positive (tensile) when θ is between 0° and 90° and negative (compressive) between 90° and 180°. Maximum values being at 45° (= τ) and 135° (= $-\tau$)
- the shear stress is positive (clockwise) for $\theta < 45°$ and negative (counter- clockwise) for $\theta > 45°$ and < 135° and again positive between $\theta > 135°$ and < 180°.
- the shear stress is zero at 45° and 135° where the normal stress is maximum.

These conclusions indicate that when a body is acted upon by pure shear stresses on two perpendicular planes, the planes inclined at 45° are subjected to a tensile stress of magnitude equal to that of the shear stress while the planes inclined at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes.

Compare this result with Eq. 2.12.

(iv) Bi-axial and Shear Stresses Condition

Let an element of a body be acted upon by two tensile stresses alongwith shear stresses acting on two perpendicular planes of the body as shown in Fig. 2.5. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively.

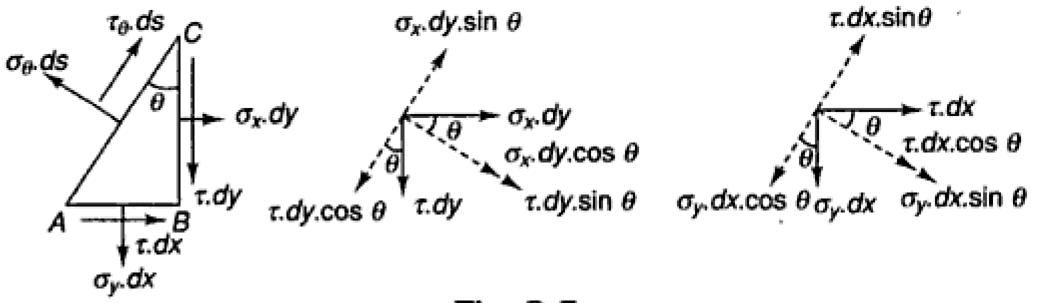


Fig. 2.5

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta ds - \sigma_x .dy.\cos\theta - \sigma_y .dx.\sin\theta - \tau.dy.\sin\theta - \tau.dx.\cos\theta$$

or
$$\sigma_\theta = \frac{\sigma_x dy\cos\theta}{ds} + \frac{\sigma_y dx\sin\theta}{ds} + \frac{\tau.dy\sin\theta}{ds} + \frac{\tau.dx\cos\theta}{ds}$$

$$= \frac{\sigma_x dy\cos\theta}{dy/\cos\theta} + \frac{\sigma_y dx\sin\theta}{dx/\sin\theta} + \frac{\tau dy\sin\theta}{dy/\cos\theta} + \frac{\tau\, dx\cos\theta}{dx/\sin\theta}$$

$$= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta$$

$$= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau\sin 2\theta \qquad (2.19)$$

$$= \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) + \tau.\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \qquad (2.20)$$

Resolving the forces in the direction of τ,

$$\tau_\theta.ds + \sigma_x.dy.\sin\theta - \sigma_y.dx.\cos\theta - \tau.dy.\cos\theta + \tau.dx.\sin\theta = 0$$

or
$$\tau_\theta = -\frac{\sigma_x dy\sin\theta}{ds} + \frac{\sigma_y dx\cos\theta}{ds} + \frac{\tau.dy\cos\theta}{ds} - \frac{\tau.dx\sin\theta}{ds}$$

$$= -\frac{\sigma_x dy\sin\theta}{dy/\cos\theta} + \frac{\sigma_y dx\cos\theta}{dx/\sin\theta} + \frac{\tau dy\cos\theta}{dy/\cos\theta} - \frac{\tau dx\sin\theta}{dx/\sin\theta}$$

$$= -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau\cos^2\theta - \tau\sin^2\theta$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad (2.21)$$

Equations 2.19, 2.20 and 2.21 can be used to determine the stresses on any inclined plane in a material under general state of stress.

To determine the planes having maximum and minimum values of direct stress, differentiate Eq. 2.20 with respect to θ and equate to zero, i.e.

$$\frac{d\sigma_\theta}{d\theta} = 0 - \frac{1}{2}(\sigma_x - \sigma_y)2\sin 2\theta + 2\tau.\cos 2\theta = 0$$

or
$$\frac{1}{2}(\sigma_x - \sigma_y)2\sin 2\theta = 2\tau.\cos 2\theta$$

or
$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \qquad (2.22)$$

This equation provides two values of 2θ differing by 180° or θ by 90°, the planes along which the direct stresses have the maximum and minimum values.

Note that the same values of θ are also obtained by equating τ_θ to zero, which indicates that shear stress is zero or does not exist on these planes. Thus it is concluded that shear stresses are zero on the planes with maximum or minimum values of direct stress (They are known as *principal planes*, to be discussed in the next section).

If σ_x and σ_y are not alike, i.e. if one of them is compressive (say σ_y is compressive), corresponding expressions can be obtained by replacing σ_y with $-\sigma_y$.

Note that in general

- As the material as a whole is in equilibrium under the action of external forces and internal resistances, an element of any shape at any point in a material will also be in equilibrium under the internal or external forces.
- An element of any shape may be considered for force analysis. Usually, the choice is made depending upon the requirements. For example, if the stresses on longitudinal and transverse axes are required, a rectangular element is a suitable choice whereas if the stresses on some inclined plane are to be found, then a triangular element has to be preferred.
- Relations derived above for various cases are valid when
  - inclination is measured in the clockwise direction with the vertical plane,

- compressive stresses are taken negative and
- the direction of the shear stresses is clockwise on the vertical planes and counter-clockwise on the horizontal planes.

In case, these parameters are chosen differently, relations have to be modified as below:

- If angle is measured counter-clockwise with the vertical planes (Fig. 2.2*b*), θ is to be replaced by $-\theta$ and the relations for direct and shear stresses for a complex system of stresses will be

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau.\sin 2\theta \qquad (2.23)$$

and $$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad (2.24)$$

- If angle is taken counter-clockwise with the horizontal plane, θ is replaced with $(90^\circ - \theta)$ and the relations are

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta + \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \qquad (2.25)$$

and $$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta \qquad (2.26)$$

- If angle is taken clockwise with the horizontal plane, θ is replaced by $[90^\circ - (-\theta)]$ or by $(90^\circ + \theta)$ and

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau.\sin 2\theta \qquad (2.27)$$

and $$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta \qquad (2.28)$$

- If the direction of the shear stresses is counter-clockwise on the vertical planes and clockwise on the horizontal planes replace τ with $-\tau$ and the relations for the general case will be

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau.\sin 2\theta \qquad (2.29)$$

and $$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta \qquad (2.30)$$

Thus, to use the derived relations directly, it must be ensured that the parameters are taken in the proper way.

2.3 PRINCIPAL STRESSES

In general, a body may be acted upon by direct stresses and shear stresses. However, it will be seen that even in such complex systems of loading, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal. These are known as *principal planes* and the normal stress across these planes as *principal stresses*. Larger of the two stresses, σ_1 is called the *major principal stress* and the smaller one σ_2 as the *minor principal stress*. The corresponding planes are known as *major* and *minor principal planes*. In two-dimensional problems, the third principal stress is taken to be zero.

Thus in principal planes shear stress is zero, i.e.

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta = 0 \qquad \text{(Eq. 2.21)}$$

or
$$\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau\cos 2\theta$$

or
$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \qquad (2.31)$$

which provides two values of 2θ differing by 180° or two values of θ differing by 90°. Thus the two principal planes are perpendicular to each other. (Also refer Eq. 2.22)

From Fig. 2.6,

$$\sin 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$

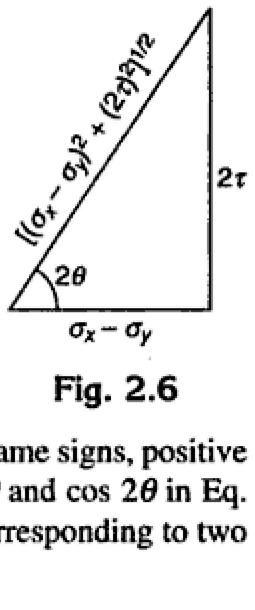


Fig. 2.6

Right-hand sides of both the above equations should have the same signs, positive or negative while using them. Substituting these values of sin 2θ and cos 2θ in Eq. 2.20, two values of the direct stresses, i.e. of principal stresses corresponding to two values of 2θ are obtained.

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} \pm \tau.\frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \qquad (2.32)$$

Maximum (Principal) Shear Stress

In any complex system of loading, the maximum and the minimum normal stresses are the principal stresses and the shear stress is zero in their planes. To find the maximum value of shear stress and its plane in such a system, consider the equation for shear stress in a plane, i.e.

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad \text{(Eq. 2.21)}$$

For maximum value of τ_θ, differentiate it with respect to θ and equate to zero,

$$\frac{d\tau_\theta}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau\sin 2\theta = 0$$

or $$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau} \qquad (2.33)$$

This indicates that there are two values of 2θ differing by 180° or two values θ differing by 90°. Thus maximum shear stress planes lie at right angles to each other.

Now, as $\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau}$ can be represented as shown in Fig. 2.7

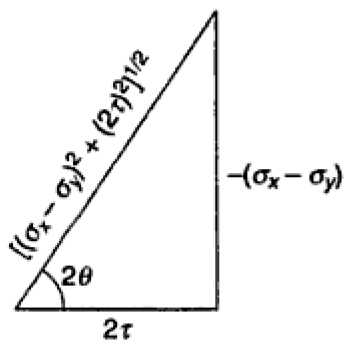


Fig. 2.7

$$\sin 2\theta = \mp\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}};$$

$$\cos 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$

Right-hand sides of both the above equations should have the opposite signs, if one is positive the other is negative while using them. Substituting these values of $\sin 2\theta$ and $\cos 2\theta$ in Eq. 2.21, two values of the shear stress are obtained.

$$\therefore \quad \tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta$$

$$= -\left(\mp\frac{1}{2}(\sigma_x - \sigma_y)\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}\right) \pm \tau.\frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

This provides maximum and minimum values of shear stress, both numerically equal. In fact the negative or minimum value indicate that it is at right angle to the positive value as discussed above and two are the complimentary shear stresses. Thus magnitude of the maximum or principal shear stress is given by

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

As maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (i)

and minimum principal stress, $\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (ii)

Subtracting (ii) from (i), $\sigma_1 - \sigma_2 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$;

$$\therefore \qquad \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

Thus in general, $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (2.34)

Now, principal planes are given by, $\tan 2\theta_p = \frac{2\tau}{\sigma_x - \sigma_y}$

and planes of maximum shear stress, $\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau}$

Multiplying the two, $\tan 2\theta_p . \tan 2\theta_s = -1$ which means

$$2\theta_s = 2\theta_p + 90^o \text{ or } \theta_s = \theta_p + 45^o$$

This indicates that the planes of maximum shear stress lie at 45° to the planes of principal axes.

The above conclusions can also be drawn from the fact that the case of biaxial stresses on a rectangular element discussed in the previous section is a case of principal stresses, as no shear stress is acting on the horizontal or vertical planes. Thus, σ_x and σ_y also denote principal stresses in the element and as in case of biaxial stresses, the maximum value of shear stress lies in the planes at 45° to the principal planes and is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

2.4 MOHR'S STRESS CIRCLE

The stress components on any inclined plane can easily be found with the help of a geometrical construction known as *Mohr's stress circle.*

Two Perpendicular Direct Stresses

Let the material of a body at a point be subjected to two like direct tensile stresses σ_x and σ_y ($\sigma_x > \sigma_y$), on two perpendicular planes *AD* and *AB* respectively (Fig. 2.8).

Make the following constructions:

- On the x-axis, take $OF = \sigma_x$ and $OE = \sigma_y$ to some scale. A stress is taken towards the right of the origin *O* (positive) if tensile and toward left (negative) if compressive.

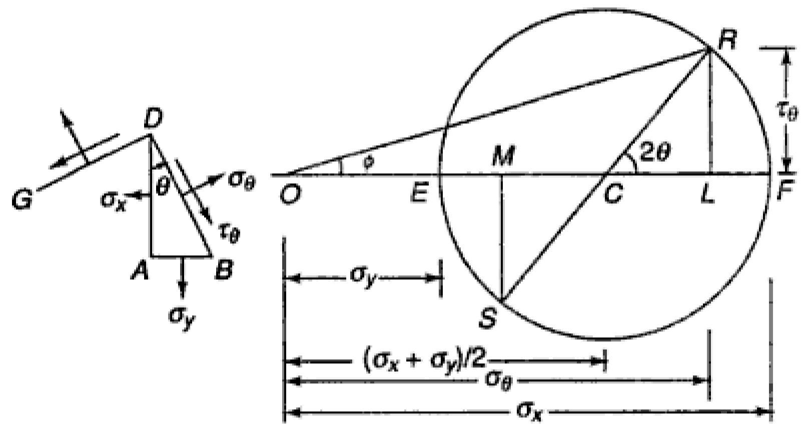


Fig. 2.8

- Bisect *EF* at *C*.
- With *C* as centre and *CE* (= *CF*) as radius, draw a circle.
  The radius *CF* represents the plane *AD* (of direct stress σ_x) and *CE*, the plane *AB* (of direct stress σ_y). Note that the two planes *AD* and *AB* which are at 90° are represented at 180° apart (or at double the angle) in the Mohr's circle. This indicates that any angular position of a plane can be located at double the angle from a particular plane.
- Locate an inclined plane in this circle by marking a radial line at double the angle at which the required plane is inclined with a given plane, e.g., if the plane *BD* is inclined at angle θ with plane *AD* in the counter-clockwise direction, then mark radius *CR* at an angle 2θ with *CF* in the counter-clockwise direction.
- Draw $LR \perp x$-axis. Join *OR*.

Now, it can be shown that *OL* and *LR* represent the normal and the shear stress components on the inclined plane *BD*.

From the geometry of the figure,

$$OC = \frac{1}{2}(OC + OC) = \frac{1}{2}[(OF - CF) + (OE + CE)]$$

$$= \frac{1}{2}[(OF - CF) + (OE + CF)] \qquad ...(CE = CF)$$

$$= \frac{1}{2}(OF + OE) = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$CL = CR \cos 2\theta = CF \cos 2\theta = \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \quad ...(CR = CF)$$

Thus $OL = OC + CL = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta = \sigma_\theta$

...(Refer Eq. 2.6)

And $LR = CR \sin 2\theta = CF \sin 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta = \tau_\theta$ ...(Refer Eq. 2.7)

The resultant of *OL* and *LR* is represented by *OR* at an angle φ with *OL*, i.e. with the direction of σ_θ. Thus the components *OL* and *RL* represent the normal and shear stress components on the plane *BD*.

Note the following:

- Direct stress component on the inclined plane *BD* represented by *OR* is on the right side of the origin, it is positive or tensile.
- Shear stresses giving a clockwise rotation are assumed positive and are above the *x*-axis. In the present case, the shear component *LR* represents a clockwise direction.
- The stress components on a plane *DG* perpendicular to *BD* are obtained by rotating the radial line *CR* through double the angle, i.e. 180° in clockwise or counter-clockwise direction. Thus *CS* represents the plane *DG*. *OM* indicates the tensile component and *SM*, the shear component.

Two Perpendicular Direct Stresses with Simple Shear

In the above-discussed case, *CR* and *CS* represent two perpendicular planes having direct tensile stresses *OL* and *OM* and shear stresses *LR* (clockwise) and *MS* (= *LR*, counter-clockwise) respectively. Now, if these happen to be the known stresses on two perpendicular planes, then stresses on any other inclined plane can easily be found by locating that plane relative to any of these planes.

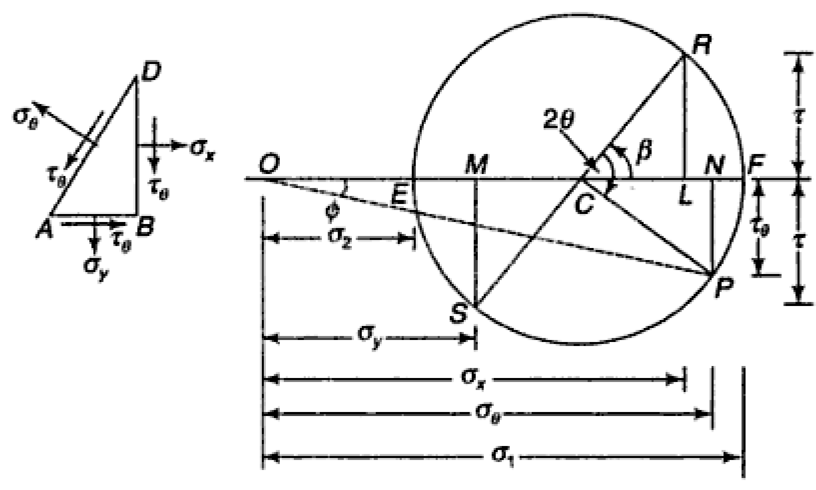


Fig. 2.9

Let *CR* and *CS* represent two perpendicular planes *BD* and *AB* respectively so that $OL = \sigma_x$, $OM = \sigma_y$ and *LR* and *MS* each equal to τ in the clockwise and counter-clockwise directions respectively (Fig. 2.9). Now if it is desired to find stresses on an inclined plane at angle θ clockwise with plane *BD*, a radial line *CP* may be drawn at angle 2θ in the clockwise direction with *CR*. Then *ON* and *NP* will represent the direct and shear components respectively on the plane *AD* and the resultant is given by *OP*.

Thus the procedure may be summarised as follows:

- Take *OL* and *OM* as the direct components of the two perpendicular stresses σ_x and σ_y.
- At *L* and *M*, draw ⊥s *LR* and *MS* on the *x*-axis each equal to τ using the same scale as for the direct stresses. For the stress system shown in Fig. 2.8, *LR* is taken upwards as the direction on plane *BD* is clockwise and *MS* downwards as the direction on plane *AB* is counter-clockwise.

- Bisect LM at C and draw a circle with C as centre and radius equal to CR ($= CS$). Let $\angle LCR = \beta$.
- Rotate the radial line CR through angle 2θ in the clockwise direction if θ is taken clockwise and let it take the position CP.
- Draw $NP \perp$ on the x-axis. Join OP.

It can be proved that ON and NP represent the normal and the shear stress components on the inclined plane AD.

From the geometry of the figure,

$$OC = \frac{1}{2}(\sigma_x + \sigma_y) \text{ as before.}$$

$$CN = CP\cos(2\theta - \beta)$$

$$= CR\cos(2\theta - \beta) \qquad ...(CP = CR)$$

$$= CR(\cos 2\theta \cos\beta + \sin 2\theta \sin\beta)$$

$$= (CR\cos\beta)\cos 2\theta + (CR\sin\beta)\sin 2\theta$$

$$= CL\cos 2\theta + LR\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \qquad ...(CL = OL - OM)$$

Thus $ON = OC + CN = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta = \sigma_\theta$ ...(Eq. 2.20)

and

$$NP = CP\sin(2\theta - \beta) = CR\sin(2\theta - \beta)$$

$$= CR(\sin 2\theta\cos\beta - \cos 2\theta \sin\beta)$$

$$= (CR\cos\beta)\sin 2\theta - (CR\sin\beta)\cos 2\theta$$

$$= CL\sin 2\theta - LR\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta = -\tau_\theta \qquad ...(\text{Eq. 2.21})$$

As NP is below the x-axis, therefore, the shear stress is negative or counter-clockwise.

Mathematically, $NP = -\left[\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta\right]$

$$= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta]$$

Principal Stresses

As shear stress is zero on principal planes, OF represents the major principal plane with maximum normal stress. In a similar way, OE represents the minor principal plane.

$$OF = OC + CF = OC + CR = OC + \sqrt{CL^2 + LR^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + \tau^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

= Major principal stress

$$OE = OC - CE = OC - CR = OC - \sqrt{CL^2 + LR^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{\{(\sigma_x - \sigma_y)^2 + 4\tau^2\}}$$

= Minor principal stress

The angles of inclination of planes of major and minor principal stresses are $\beta/2$ and $(90° + \beta/2)$ respectively clockwise with the plane of stress σ_x.

Example 2.1 *Two pieces of wood of section 50 mm × 30 mm are joined together along a plane at 60° with the x-axis. If the strength of the joint is 7.5 MPa in tension and 4 MPa in shear, determine the maximum force which the member can sustain.*

Solution Let the maximum stress along the x-axis be σ_x.

Angle with the vertical plane = 90° – 60° = 30°

Then at the joint, $\sigma_\theta = \sigma_x \cos^2 \theta = \sigma_x \cos^2 30° = \frac{3}{4}\sigma_x$

or $$7.5 = \frac{3}{4}.\sigma_x \quad \text{or} \quad \sigma_x = 10 \text{ MPa} \qquad \text{(i)}$$

$$\tau_\theta = -\frac{1}{2}\sigma_x \sin 2\theta = -\frac{1}{2}\sigma_x \sin 60° = -\frac{\sqrt{3}}{4}\sigma_x.$$

or $$-4 = -\frac{\sqrt{3}}{4}\sigma_x \quad \text{or} \quad \sigma_x = 9.24 \text{ MPa} \qquad \text{(ii)}$$

(shear stress at the joint is assumed counter-clockwise to have positive axial tensile stress σ_x)

From (i) and (ii), for safety of the joint, the maximum axial stress to be taken by the member is 9.24 MPa.

Maximum force, $P = (50 \times 30) \times 9.24 = 13\,860$ N or 13.86 kN

Example 2.2 *A rectangular block is subjected to a two perpendicular stresses of 10 MPa tension and 10 MPa compression. Determine the stresses on planes inclined at (i) 30° (ii) 45° and (iii) 60° with the x-axis.*

Solution

(i) Inclination with the vertical plane = 90° – 30° = 60°

$$\sigma_{60} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta \qquad \text{(Eq. 2.5)}$$

$$= 10(\cos 60° - \sin 60°) = 10\left(\frac{1}{4} - \frac{3}{4}\right) = -5 \text{ MPa (compression)}$$

or using the expression,

$$\sigma_{60} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \qquad \text{(Eq. 2.6)}$$

$$= \frac{1}{2}(10 + (-10) + \frac{1}{2}[10 - (-10)]\cos 120° = -5 \text{ MPa (compression)}$$

$$\tau_{60} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10-(-10)]\sin 120^\circ = -8.66 \text{ MPa (ccw)}$$

Solution by Mohr's circle is shown in Fig. 2.10. Adopt the following procedure:

- Take *OF* equal to 10 MPa to some suitable scale to the right of *O* for tensile stress. Similarly, take *OE* to the left for compressive stress.
- The centre of *EF* is the centre of the Mohr's circle. Draw the circle passing through *E* and *F* points. Now *OF* and *OE* represent the planes of tensile and compressive stresses respectively.
- Make angle $FCR = 120^\circ$, i.e double the angle of the inclined plane with *OF* in the clockwise direction.

Then *CR* represents the inclined plane.

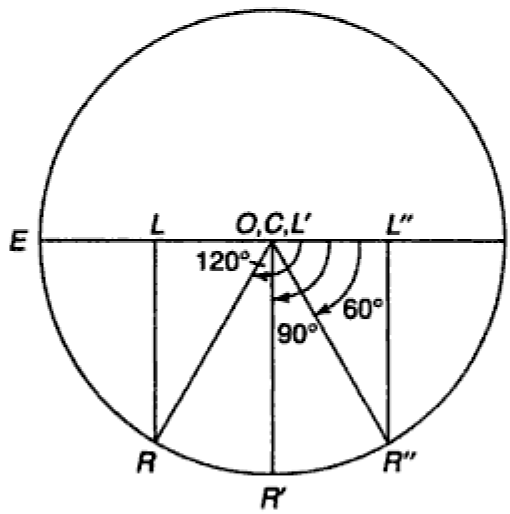


Fig. 2.10

$\sigma_{60} = OL = 5$ MPa (compressive being on the left of point *O*)
$\tau_{60} = LR = 8.66$ MPa (ccw being below the *x*-axis)

(ii) Inclination with the vertical plane $= 90^\circ - 45^\circ = 45^\circ$

$$\sigma_{45} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$= 10(\cos^2 45^\circ - \sin^2 45^\circ) = 10\left(\frac{1}{2}-\frac{1}{2}\right) = 0$$

and

$$\tau_{45} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10-(-10)]\sin 90^\circ = -10 \text{ MPa (ccw)}$$

Solution by Mohr's circle is self-explanatory.

(iii) Inclination with the vertical plane $= 90^\circ - 60^\circ = 30^\circ$

$$\sigma_{60} = 10(\cos^2 30^\circ - \sin^2 30^\circ) = 10\left(\frac{3}{4}-\frac{1}{4}\right) = 5 \text{ MPa}$$

and

$$\tau_{60} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10-(-10)]\sin 60^\circ = -8.66 \text{ MPa (ccw)}$$

Solution by Mohr's circle is self-explanatory.

Example 2.3 *A piece of material is subjected to two perpendicular stresses as follows:*

(a) tensile stresses of 100 MPa and 60 MPa

(b) tensile stress of 100 MPa and compressive stress of 60 MPa
(c) compressive stress of 100 MPa and tensile stress of 60 MPa
(d) compressive stresses of 100 MPa and 60 MPa.

Determine normal and tangential stresses on a plane inclined at 30° to the plane of 100 MPa stress. Also find the resultant and its inclination with the normal stress.

Solution

(a) Inclination with the vertical plane = 30°

$$\sigma_{30} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta = 100\cos^2 30^\circ + 60\sin^2 30^\circ$$

$$= 100\times\frac{3}{4} + 60\times\frac{1}{4} = 90 \text{ MPa (tensile)}$$

$$\tau_{30} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}(100-60)\sin 60^\circ = -17.32 \text{ MPa (ccw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{90^2 + (-17.32)^2} = 91.65 \text{ MPa}$$

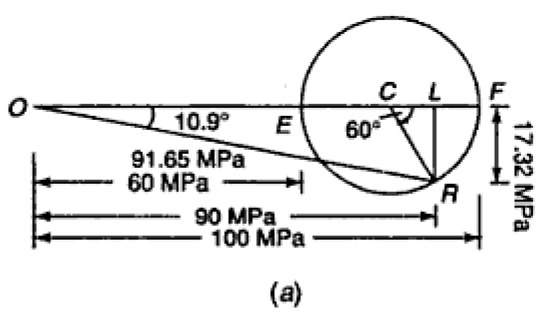
Its inclination with normal stress σ_{30}

$$\tan\varphi = \frac{\tau_{30}}{\sigma_{30}} = \frac{17.32}{90} = 0.1924 \quad \text{or} \quad \varphi = 10.89^\circ$$

The inclination of the resultant with σ_x can also be found,

$$\tan\alpha = \frac{\sigma_y}{\sigma_x}\tan 30^\circ = \frac{60}{100}\tan 30^\circ = 0.346 \quad \text{or} \quad \alpha = 19.11^\circ$$

Thus $\alpha + \varphi = 19.11^\circ + 10.89^\circ = 30^\circ = \theta$



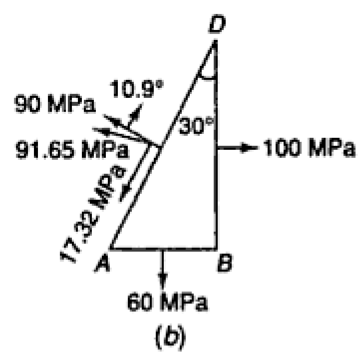


Fig. 2.11

Solution by Mohr's circle is shown in Fig. 2.11 (*a*). The procedure is as follows:

- Take *OF* and *OE* equal to 100 MPa and 60 MPa respectively to some suitable scale to the right of *O* for tensile stresses.
- The centre of *EF* is the centre of the Mohr' circle. Draw the circle passing through *E* and *F* points. Now *OF* and *OE* represents the planes of tensile stresses 100 MPa and 60 MPa respectively.

- Make angle $FCR = 60°$, i.e. double the angle of the inclined plane with OF in the clockwise direction.

Then CR represents the inclined plane.

$\sigma_{30} = OL = 90$ MPa (tensile)

$\tau_{30} = LR = 17.32$ MPa (counter-clockwise)

$\sigma_r = OR = 91.65$ MPa

Inclination of the resultant with OL or σ_{30}, $\varphi = 10.9°$

The results are shown in Fig. 2.11(*b*).

(b) $$\sigma_{30} = 100\cos^2 30° - 60\sin^2 30° = 100 \times \frac{3}{4} - 60 \times \frac{1}{4} = 60 \text{ MPa (tensile)}$$

$$\tau_{30} = -\frac{1}{2}[100 - (-60)]\sin 60° = -80 \times 0.866 = -69.28 \text{ MPa (ccw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{60^2 + (-69.28)^2} = 91.65 \text{ MPa}$$

inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.11°$

α can be found to be $-19.11°$.

Solution by Mohr's circle is shown in Fig. 2.12 which is self-explanatory.

$\sigma_{30} = OL = 60$ MPa (tensile)

$\tau_{30} = LR = 69.3$ MPa (counter-clockwise)

$\sigma_r = OR = 91.65$ MPa

Inclination of the resultant with OL or σ_{30}, $\varphi = 49.1°$

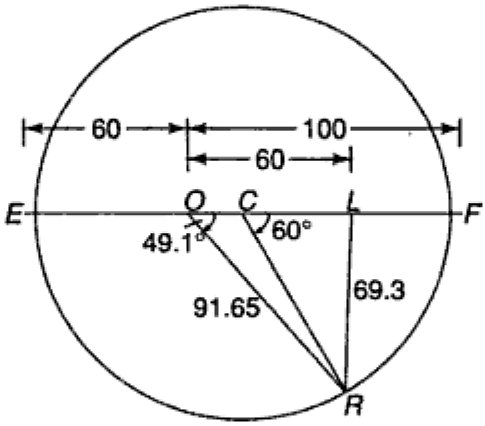


Fig. 2.12

(c) $$\sigma_{30} = -100\cos^2 30° + 60\sin^2 30°$$
$$= -100 \times \frac{3}{4} + 60 \times \frac{1}{4}$$
$$= -60 \text{ MPa (comp.)}$$

$$\tau_{30} = -\frac{1}{2}(-100 - 60)\sin 60° = 80 \times 0.866 = 69.28 \text{ MPa (cw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{(-60)^2 + (69.28)^2} = 91.65 \text{ MPa}$$

Inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.11°$

α can be found to be $-19.11°$.

(d) $$\sigma_{30} = -100\cos^2 30° - 60\sin^2 30°$$
$$= -100 \times \frac{3}{4} - 60 \times \frac{1}{4} = -90 \text{ MPa (comp.)}$$

$$\tau_{30} = -\frac{1}{2}[-100-(-60)]\sin 60^{\circ} = 20 \times 0.866 = 17.32 \text{ MPa (cw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{(-90)^2 + (17.32)^2} = 91.65 \text{ MPa}$$

Inclination with σ_{30}, $\tan\varphi = \frac{\tau_{30}}{\sigma_{30}} = \frac{17.32}{90} = 0.1924$

or $\varphi = 10.89^{\circ}$

α can be found to be -19.11°.

Example 2.4 *A piece of material is subjected to two perpendicular tensile stresses of 100 MPa and 60 MPa. Determine the plane on which the resultant stress has maximum obliquity with the normal. Also find the resultant stress on this plane.*

Solution For maximum obliquity of the resultant with the normal to a plane is given by

$$\tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}} = \sqrt{\frac{100}{60}} = 1.29 \quad \text{or} \quad \theta = 52.24^{\circ} \qquad \text{...(Eq. 2.10)}$$

Direct stress,

$$\sigma_{52.24} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$
$$= 100\cos^2 52.24^{\circ} + 60\sin^2 52.24^{\circ}$$
$$= 100 \times 0.375 + 60 \times 0.735 = 37.5 + 37.5 = 75 \text{ MPa}$$

Shear stress,

$$\tau_{52.24} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$
$$= -\frac{1}{2}(100-60)\sin 104.48^{\circ} = -19.365 \text{ MPa (ccw)}$$

Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{75^2 + 19.365^2} = 77.46 \text{ MPa}$$

$$\tan\varphi = \frac{19.365}{75} = 0.2582 \quad \text{or} \quad \varphi = 14.48^{\circ}$$

The resultant and its inclination can also be found directly by using relations

$$\sigma = \sqrt{\sigma_x^2 \cos^2\theta + \sigma_x^2 \sin^2\theta}$$
$$= \sqrt{100^2\cos^2 52.24^{\circ} + 60^2 \sin^2 52.24^{\circ}}$$
$$= \sqrt{3749.8 + 2250} = 77.46 \text{ MPa}$$

Inclination with σ_x,

$$\tan\alpha = \frac{60}{100}\tan 52.24^{\circ} \quad \text{or} \quad \alpha = 37.76^{\circ}$$

(Note that $\alpha + \varphi = 37.76^{\circ} + 14.48^{\circ} = 52.24^{\circ} = \theta$)

Solution by Mohr's circle is shown in Fig. 2.13. *OR* is tangent to the circle for maximum obliquity.

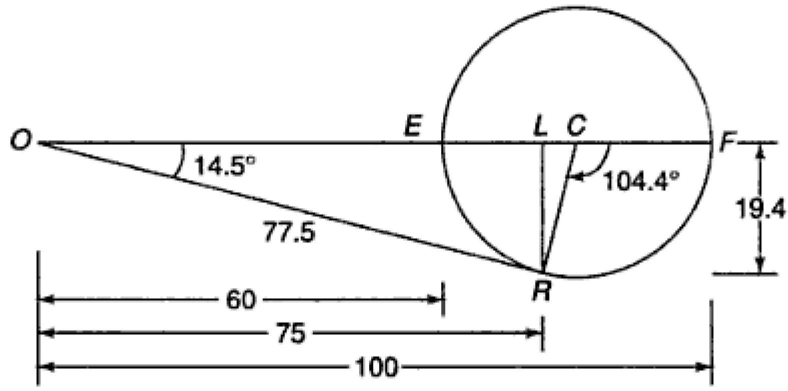


Fig. 2.13

$$\theta = 104.4/2 = 52.2^\circ$$
$$\sigma_{52.2} = OL = 75 \text{ MPa (tensile)}$$
$$\tau_{52.2} = LR = 19.4 \text{ (counter-clockwise)}$$
$$\sigma_r = OR = 77.5 \text{ MPa}$$

Inclination of the resultant with *OL* or $\sigma_{52.2}$, $\varphi = 14.5^\circ$

Example 2.5 *The stresses on two perpendicular planes through a point in a body are 30 MPa and 15 MPa both tensile alongwith shear stress of 25 MPa. Find*

(i) the magnitude and direction of principal stresses

(ii) the planes of maximum shear stress

(iii) the normal and shear stresses on the planes of maximum shearing stress.

Solution

$$\text{Principal stress} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$
$$= \frac{1}{2}(30+15) \pm \frac{1}{2}\sqrt{(30-15)^2 + 4(25)^2}$$
$$= 22.5 \pm \frac{1}{2}\sqrt{225+2500} = 22.5 \pm 26.1$$
$$= 48.6 \text{ MPa (tensile)} \quad \text{and } -3.6 \text{ MPa (compressive)}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2\times 25}{30-15} = \frac{50}{15} = 3.333 \quad \text{or} \quad 2\theta = 73.3$$

or $\theta_1 = 36.65^\circ$ and $\theta_2 = 36.65^\circ + 90^\circ = 126.65^\circ$

However, to correlate the angle of the major principal plane, it is necessary to calculate the stress at one of the angles,

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$
$$= 30\cos^2 36.65^\circ + 15\sin^2 36.65^\circ + 25\sin[2(36.65^\circ)]$$
$$= 19.31 + 5.345 + 23.946 = 48.6 \text{ MPa}$$

Thus major principal stress 48.6 MPa occurs on the plane inclined at 36.65° with the plane of 30 MPa tensile stress (clockwise) and the minor stress occurs on plane at 126.65°.

Maximum shear stress occurs on planes at 45° to the principal planes, i.e. on planes at 36.65° + 45° = 81.65° and 126.65° + 45° = 171.65° (or at 36.65° – 45° = – 8.35° which is the same plane as 171.65°)

$$\sigma_{81.65} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$
$$= 30\cos^2 81.65^\circ + 15\sin^2 81.65^\circ + \tau \sin 2(81.65^\circ)$$
$$= 0.632 + 14.684 + 7.184 = 22.5 \text{ MPa}$$

On the plane at –8.35°, the stress value will also be found to be same.

$$\tau_{max} = \pm \frac{1}{2}(48.6 - (-3.6) = \pm 26.1 \text{ MPa}$$

Solution by Mohr's circle is shown in Fig. 2.14 which is self-explanatory.

In brief, OL = 30 MPa, OM = 15 MPa, LR = MS = 25 MPa. Mohr's circle is drawn with C, the mid-point of LM as centre and passing through R and S.

- Major principal stress = OF = 48.6 MPa (tensile) at angle 73/2 = 36.5° clockwise of CR or plane of 30 MPa tensile stress.
- Minor principal stress = OE = 3.6 MPa (compressive) at angle 253/2 = 126.5° clockwise of CR or plane of 30 MPa tensile stress. Maximum shear stress = CG = CH = 26.1 MPa

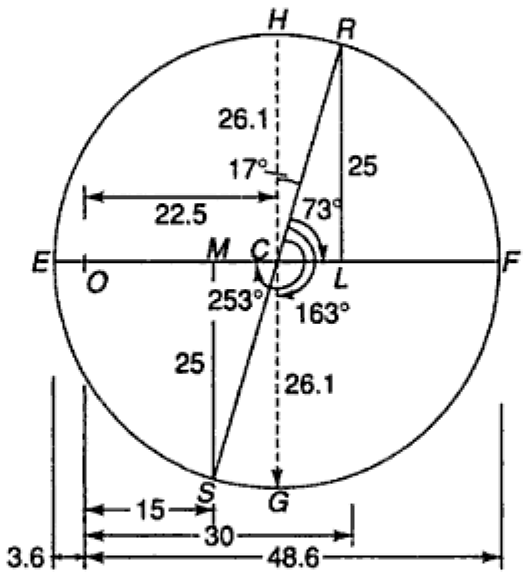


Fig. 2.14

- Inclination of the planes of maximum shear stresses: 163/2 = 81.5° clockwise of CR or plane of 30 MPa tensile stress and (343/2 = 171.5°) clockwise or 17/2 = 8.5° counter-clockwise of CR or plane of 30 MPa tensile stress.

$$\sigma_{81.65} = OC = 22.5 \text{ MPa}$$

Example 2.6 *In a 2-D stress system, stresses at a point in a material are 50 MPa compression and 30 MPa shearing in one plane and 20 MPa tensile and a shearing stress in another plane at 60° to the first one. Determine the value of the shearing stress in the second plane and the principal stresses and position of their planes.*

Solution Let σ_y be the second tensile stress in a plane perpendicular to that of σ_x. Then in any other plane at an angle θ to the first one,

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \qquad \text{(Eq. 2.20)}$$

Thus in a plane at 60°,

$$20 = \frac{1}{2}(-50+\sigma_y)+\frac{1}{2}(-50-\sigma_y)\cos 120^\circ + 30\sin 120^\circ$$

$$40 = -50+\sigma_y + (-50-\sigma_y)(-0.5)+60\times 0.866$$

$$40 = -50+\sigma_y + 25 + 0.5\sigma_y + 51.96$$

$$1.5\,\sigma_y = 13.04 \quad \text{or} \quad \sigma_y = 8.69 \text{ MPa}$$

$$\tau_\theta = -(-50-8.69)\sin 120^\circ + 30\cos 120^\circ = 10.414 \text{ MPa}$$

$$\text{Principal stress} = \frac{1}{2}(\sigma_x+\sigma_y)\pm\frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}$$

$$= \frac{1}{2}(-50+8.69)\pm\frac{1}{2}\sqrt{(-50-8.69)^2+4(30)^2}$$

$$= -20.66 \pm 41.97 = 21.31 \text{ and } -62.63 \text{ MPa}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x-\sigma_y} = \frac{2\times 30}{-50-8.69} = -1.0223$$

or $2\theta = -45.63$ or $\theta_1 = -22.82^\circ$ or 157.18°

and $\theta_2 = -22.82^\circ + 90^\circ = 67.18^\circ$

Example 2.7 *Figure 2.15(a) shows the resultant stresses on two planes at a certain point in a material. On a certain plane it is 800 MPa compressive at an angle of 30° to its normal and on another plane it is 600 MPa tensile at an angle of 75° to its normal. Determine the angle between the planes. Also find the principal stresses and their directions to the given plane.*

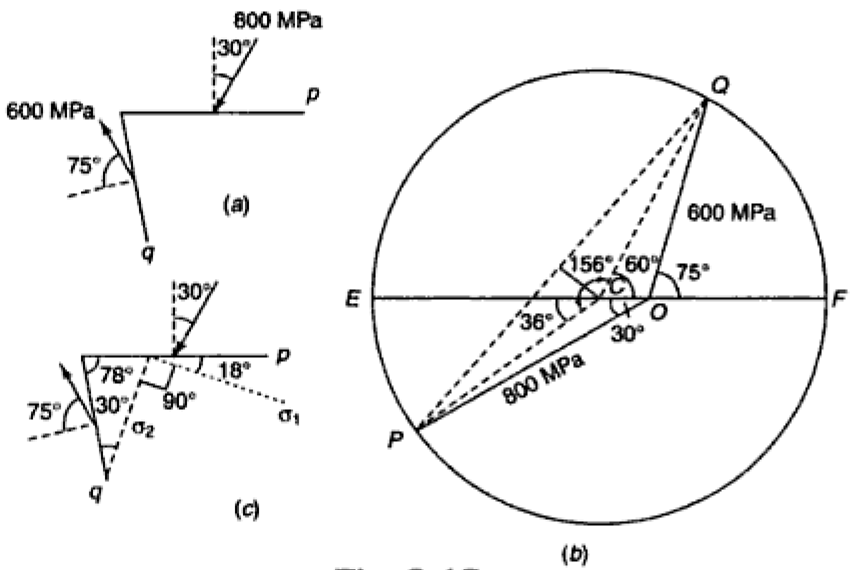


Fig. 2.15

Solution On the plane p, the resultant stress is 800 MPa compressive, its normal component will be compressive and the shear component counter-clockwise.

Thus take a radial line OP at an angle of 30° with the horizontal to the left of O downwards indicating compressive normal stress and counter-clockwise shear stress (Fig. 2.15*b*).

On the plane q, the resultant stress is 600 MPa tensile. Take a radial line OQ at an angle of 75° with the horizontal to the right of O upwards indicating tensile normal stress and clockwise shear stress.

Draw a circle passing through points P and Q and having centre on the horizontal line through O. The circle can be drawn by drawing a right bisector of PQ, the intersection of which with the horizontal line through O is the point C. Now, Mohr circle is drawn with centre C and radius equal to CP or CQ.

Angle between the planes = $\angle PCQ/2 = 156°/2 = 78°$

Major principal stress = OE = 825 MPa (compressive) at an angle equal to half of $\angle ECP$, i.e 36/2 = 18° clockwise of plane p.

Minor principal stress = OF = 525 MPa (tensile) at an angle equal to half of $\angle FCQ$, i.e 60/2 = 30° clockwise of plane q.

The principal planes are shown in Fig. 2.15*c*.

Example 2.8 *Figure 2.16(a) shows the stresses at a point in a material subjected to 2-D stresses. The stresses on a certain plane are 90 MPa tensile and 40 MPa shear whereas in another plane 60 MPa tensile and 30 MPa shear. Determine the angle between the planes. Also find the magnitudes and directions of the principal planes.*

Solution The Mohr's circle is self-explanatory, as shown in Fig. 2.16*b*.

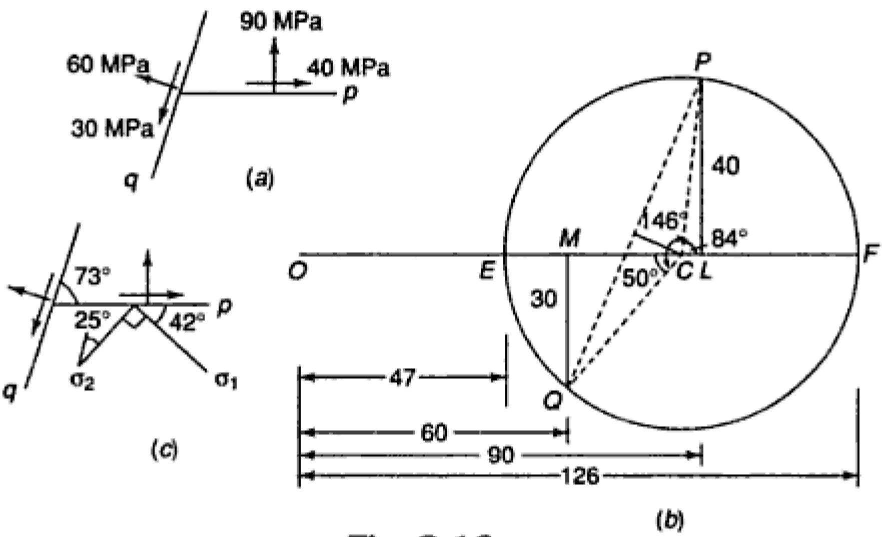


Fig. 2.16

Take OL = 90 MPa ; LP = 40 MPa; OM = 60 MPa; MQ = 30 MPa

Join PQ. Draw right bisector of PQ intersecting OL in C. With C as centre draw a circle passing through P and Q.

Angle between the planes = $\angle PCQ/2 = 146°/2 = 73°$

Major principal stress = OF = 126 MPa (tensile) at 84°/2 or 42° clockwise of plane p

Minor principal stress = OE = 47 MPa (tensile) at 50°/2 clockwise of plane q

The principal planes are shown in Fig. 2.15(*c*).

2.5 THREE COPLANAR STRESSES

Referring to Fig. 2.17, the radial lines *CR*, *CP* and *CS* in Mohr's circle represent three planes *BD*, *AD* and *AB* respectively. The normal stresses on these planes are *OL*, *ON* and *OM* respectively and lie in a plane (the surface of the paper). The angle between any two planes in Mohr's circle is twice the actual angle between the planes. This property of Mohr's circle is made use when three direct stresses alongwith the angular positions of their planes are known.

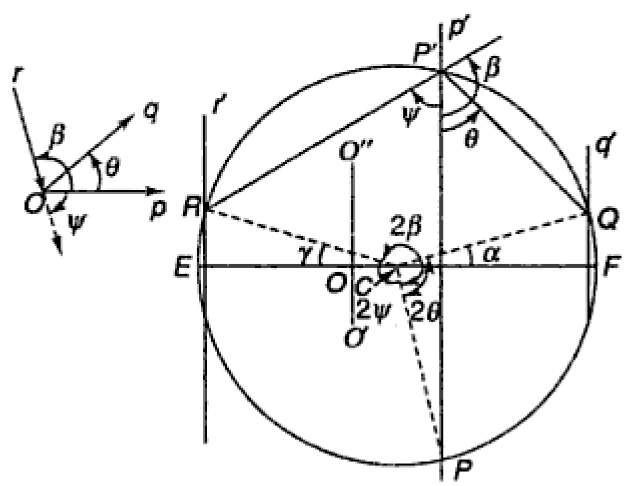


Fig. 2.17

The procedure is as follows:

- Mark a point O'. Draw a vertical line $O'O''$ through it (Fig. 2.17).
- Draw three lines p', q' and r' parallel to the vertical line through O' at distances representing the direct stresses in the directions *p*, *q* and *r* to a suitable scale. Tensile stresses (positive) are taken on the right side of $O'O''$ whereas the compressive stresses (negative) on the left. Assuming that the stresses in the directions *p* and *q* are tensile and in the direction *r* it is compressive, the lines p', q' are taken towards right, and r' towards left.
- Take a point at a convenient position on the middle vertical line. Assume that the middle line is p' and the point taken is P'.
- Draw a line through P' making an angle θ with the vertical through P' in the same direction sense as the plane *q* has with the plane *p*. In this case it is taken counter-clockwise. Let this line intersect with the vertical through q' at *Q*.
- In the same way, draw a line through P' making an angle β with the vertical through P' in the counter-clockwise direction or at angle ψ in the clockwise direction. Let it intersect with the vertical through r' at *R*.
- Draw a circle passing through P', *Q* and *R* by taking perpendicular bisectors of $P'Q$ and $P'R$ (not shown in the figure), the intersection locates the centre *C*. This is the Mohr's circle.
- Let the vertical line through P' have the second intersect point with the circumference of the circle at *P* as shown in the figure. Join *CP*, *CQ* and *CR*.

Now the radial lines *CP*, *CQ* and *CR* represent the planes of stresses *p*, *q* and *r* respectively because

- The angle made by plane *CQ* with the plane *CP* at the centre of the circle is the angle made by chord *PQ* at the centre which is 2θ, i.e. double of angle $PP'Q$ made by the chord *PQ* at the circumference of the circle.
- The angle made by plane *CR* with the plane *OP* at the centre is 2β counter-clockwise or 2 ψ clockwise, i.e. twice of that made by chord *PR* at P' on the circumference of the circle.

Let a horizontal line through C intersect the vertical through O' at O and the circle at E and F as shown in the above figure.

Principal stresses are given by OF and OE.

The angular position of major principal stress is $\alpha/2$ clockwise of plane CQ and of minor principal stress $\gamma/2$ counter-clockwise of plane CR.

Example 2.9 *Fig. 2.18(a) shows three direct stresses in three coplanar directions p, q and r at a particular point. Determine the magnitude and the directions of the principal stresses.*

Solution Draw the Mohr's circle as follows:

- Take a vertical line $O'O''$. Draw three more vertical lines p', q' and r' parallel to line $O'O''$ at distances representing stresses of 80 MPa, 30 MPa and 50 MPa respectively to a suitable scale. Lines of 80 MPa and 30 MPa are taken to the right being tensile and 50 MPa on the left being compressive.
- Take a convenient point on the middle line q', say point Q'. Draw a line making an angle 45° through this point in the clockwise direction as plane p is at 45° of plane q in the clockwise direction. Let this line intersect line p' at point P.
- Similarly, draw a line through Q' at an angle (105° – 45°), i.e. 60° in the counter-clockwise direction as plane r' is at 60° in counter-clockwise direction of plane q. Let this line intersect line r' at R.
- Draw a circle passing through points P, Q' and R by taking right bisectors of PQ' and RQ'. Let their point of intersecting be C which is the centre of the circle.
- Let the other point of intersection of line q' with the circumference be Q. Then CP, CQ and CR represent the planes of p, θ and r respectively.

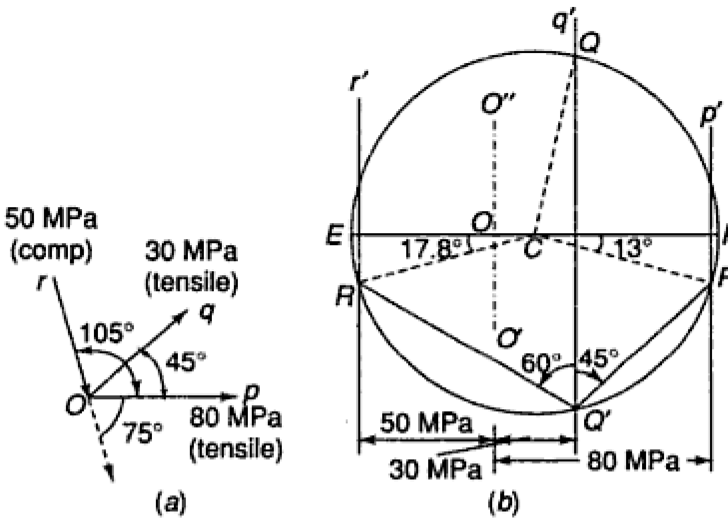


Fig. 2.18

- Draw a horizontal line through C intersecting the circle at E and F.

Major principal stress = OF = 82.5 MPa (tensile) at an angle equal to half of $\angle FCP$, i.e $13/2 = 6.5°$ counter-clockwise of p.

Minor principal stress = OE = 52.5 MPa (compressive) at an angle equal to half of $\angle ECR$, i.e $17.8 = 8.9°$ clockwise of plane r.

2.6 ELLIPSE OF STRESS

This is another graphical method to be used when a material is subjected to direct stresses σ_x and σ_y. The method is as follows:

- Draw two circles with O as centre and radii equal to σ_x and σ_y taken to a suitable scale (Fig. 2.19).
- Through O draw AB parallel to the inclined plane.
- Draw $OE \perp AB$ through O intersecting the inner circle at D and outer circle at E.
- Draw $EG \perp OX$.
- Draw $DG \perp EF$.

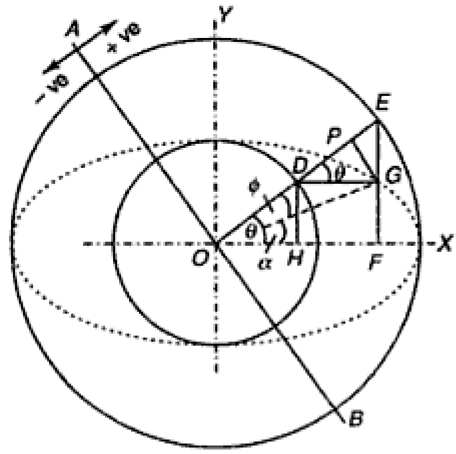


Fig. 2.19

Now, $OP = OD + DP = OD + DG\cos\theta$

$= \sigma_y + (DE\cos\theta)\cos\theta = \sigma_y + (\sigma_x - \sigma_y)\cos^2\theta$

$= \sigma_x\cos^2\theta + \sigma_y(1-\cos^2\theta) = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta = \sigma_\theta$ (Refer Eq. 2.5)

and $PG = DG\sin\theta = (DE\cos\theta)\sin\theta$

$= (\sigma_x - \sigma_y)\cos\theta\sin\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau_\theta$

(Refer Eq. 2.24 with $\tau = 0$)

Also, $OF = OE\cos\theta = \sigma_x\cos\theta$

and $FG = HD\cos\theta = \sigma_y\sin\theta$

$\therefore \quad OG = \sqrt{\sigma_x^2\cos^2\theta + \sigma_y^2\sin^2} = \sigma_r$ (Refer Eq. 2.8)

$$\tan\alpha = \frac{\sigma_y\sin\theta}{\sigma_x\cos\theta} = \frac{\sigma_y}{\sigma_x}\tan\theta \quad \text{(Refer Eq. 2.11)}$$

$$\tan\varphi = \frac{PG}{OP} = \frac{(\sigma_x - \sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta + \sigma_y\sin^2\theta} \quad \text{(Refer Eq. 2.9)}$$

For different values of θ, point G can be located, the locus of which is evidently an ellipse as shown in the figure. The diagram is thus known as ellipse of stress.

Example 2.10 *A piece of material is subjected to two perpendicular stresses as follows:*

(a) Tensile stresses of 100 MPa and 60 MPa

(b) Tensile stress of 100 MPa and compressive stress of 60 MPa

Determine normal and tangential stresses on a plane inclined at 30° to the plane of 100 MPa stress. Also find the resultant and its inclination with the normal stress using ellipse stress method.

Solution

(a) In the statement it is not specified whether the inclined plane is clockwise or counter-clockwise relative to the plane of 100 MPa stress. So, it can be taken in any sense relative to *OX*. This merely affects the sense of direction of shear stress on the inclined plane. Ellipse of stress for the given data is shown in Fig. 2.20 which is self-explanatory. The results are

$\sigma_\theta = OP = 90$ MPa;
$\tau_\theta = GP = 17.3$ MPa;
$\sigma_r = OG = 91.6$ MPa;
$\varphi = \angle POG = 10.9°$;
$\alpha = \angle FOG = 19.1°$

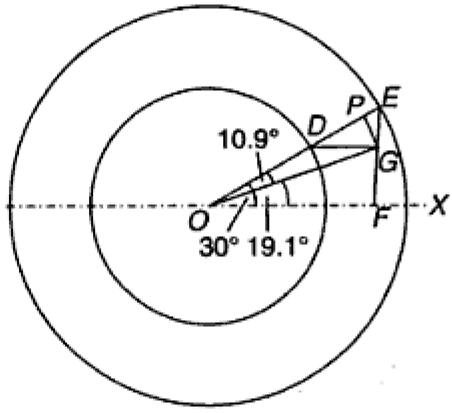


Fig. 2.20

(b) Ellipse of stress for the given data is shown in Fig. 2.21. *OE* represents the tensile stress and *OD* the compressive stress. The results are

$\sigma_\theta = OP = 60$ MPa;
$\tau_\theta = GP = 69.3$ MPa;
$\sigma_r = OG = 91.6$ MPa;
$\varphi = \angle POG = 49.1°$;
$\alpha = \angle FOG = 19.1°$

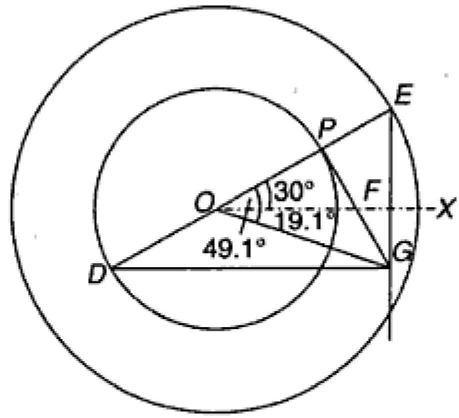


Fig. 2.21

2.7 PRINCIPAL STRESSES FROM PRINCIPLE STRAINS

(a) Two-Dimensional System

In a two-dimensional system $\sigma_3 = 0$, and

$$\varepsilon_1 = \sigma_1/E - v\sigma_2/E$$

or $\varepsilon_1.E = \sigma_1 - v\sigma_2$ or $\sigma_1 = \varepsilon_1.E + v\sigma_2$ (i)

and $\varepsilon_2 = \sigma_2/E - v\sigma_1/E$

or $\varepsilon_2.E = \sigma_2 - v\sigma_1$ (ii)

Inserting the value of σ_1 from (i) in (ii),

$$\varepsilon_2.E = \sigma_2 - v(\varepsilon_1.E + v\sigma_2) = \sigma_2(1 - v^2) - v\varepsilon_1.E$$

or $$\sigma_2 = \frac{E(v\varepsilon_1 + \varepsilon_2)}{1 - v^2} \quad (2.35)$$

Similarly, $$\sigma_1 = \frac{E(v\varepsilon_2 + \varepsilon_1)}{1 - v^2} \quad (2.36)$$

(b) Three-Dimensional System

We have

$$\varepsilon_1 = \sigma_1/E - v\sigma_2/E - v\sigma_3/E \quad \text{(i)}$$

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E \qquad \text{(ii)}$$
$$\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E \qquad \text{(iii)}$$

From (i),

$$\sigma_1 = E\varepsilon_1 + \nu(\sigma_2 + \sigma_3)$$

Inserting this value in (ii) and (iii),

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_3 - \nu[E\varepsilon_1 + \nu(\sigma_2 + \sigma_3)]$$

or $$E(\varepsilon_2 + \nu\varepsilon_1) = \sigma_2(1 - \nu^2) - \sigma_3(1 + \nu)\nu \qquad \text{(iv)}$$

Similarly,

$$E(\varepsilon_3 + \nu\varepsilon_1) = \sigma_3(1 - \nu^2) - \sigma_2(1 + \nu)\nu \qquad \text{(v)}$$

Multiplying (iv) by ν and (v) by $(1 - \nu)$,

$$E(\varepsilon_2 + \nu\varepsilon_1)\nu = \sigma_2(1 - \nu^2)\nu - \sigma_3(1 + \nu)\nu^2 \qquad \text{(vi)}$$
$$E(\varepsilon_3 + \nu\varepsilon_1)(1 - \nu) = \sigma_3(1 - \nu^2)(1 - \nu) - \sigma_2(1 - \nu^2)\nu \qquad \text{(vii)}$$

Adding (vi) and (vii),

$$E[(\varepsilon_2 + \nu\varepsilon_1)\nu + (\varepsilon_3 + \nu\varepsilon_1)(1 - \nu)] = \sigma_3(1 - \nu^2)(1 - \nu) - \sigma_3(1 + \nu)\nu^2$$
$$E(\varepsilon_2\nu + \nu\varepsilon_1\nu + \varepsilon_3 + \nu\varepsilon_1 - \nu\varepsilon_3 - \nu\varepsilon_1\nu) = \sigma_3[(1 - \nu^2)(1 - \nu) - (1 + \nu)\nu^2]$$
$$E(\varepsilon_2\nu + \varepsilon_3 + \nu\varepsilon_1 - \nu\varepsilon_3) = \sigma_3[1 - \nu^2 - \nu + \nu^3 - \nu^2 - \nu^3]$$
$$E[(1 - \nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\nu] = \sigma_3[1 - \nu^2 - \nu - \nu^2] = \sigma_3(1 + \nu)(1 - 2\nu)$$

Thus

$$\sigma_3 = \frac{E[(1-\nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\nu]}{(1+\nu)(1-2\nu)} \qquad (2.37)$$

Similarly,

$$\sigma_1 = \frac{E[(1-\nu)\varepsilon_1 + (\varepsilon_2 + \varepsilon_3)\nu]}{(1+\nu)(1-2\nu)} \qquad (2.38)$$

and

$$\sigma_2 = \frac{E[(1-\nu)\varepsilon_2 + (\varepsilon_3 + \varepsilon_1)\nu]}{(1+\nu)(1-2\nu)} \qquad (2.39)$$

2.8 STRAIN ANALYSIS

If direct and shear strains along x- and y-directions are known, the linear or normal strain (ε_θ) in a direction at angle θ with the x-direction of a body can be found by the following method:

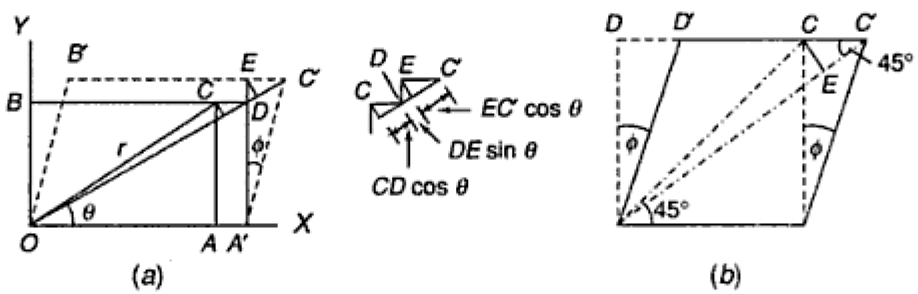


Fig. 2.22

Let a rectangular element $OACB$ with angle of the diagonal θ with the direction of ε_x or x-axis distorts to become a parallelogram under the action of linear strains ε_x, ε_y

and shear strain φ as shown in Fig. 2.22(a). Point C moves to C'. Let r be the length of the diagonal OC.

Now, $$CC' = CD \cos\theta + DE \sin\theta + EC' \cos\theta$$
$$= (\varepsilon_x.r.\cos\theta)\cos\theta + (\varepsilon_y.r.\sin\theta)\sin\theta + (\varphi.r.\sin\theta)\cos\theta$$
$$= (\varepsilon_x.r.\cos^2\theta) + (\varepsilon_y.r.\sin^2\theta) + \varphi.r.\sin\theta.\cos\theta$$

Since strain of the diagonal, $\varepsilon_\theta = CC'/r$

$$\therefore \quad \varepsilon_\theta = (\varepsilon_x.\cos^2\theta) + (\varepsilon_y.\sin^2\theta) + \varphi.\sin\theta.\cos\theta$$
$$= \frac{1}{2}\varepsilon_x(1+\cos 2\theta) + \frac{1}{2}\varepsilon_y(1-\cos 2\theta) + \frac{1}{2}\varphi\sin 2\theta$$
$$= \frac{1}{2}(\varepsilon_x+\varepsilon_y) + \frac{1}{2}(\varepsilon_x-\varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi\sin 2\theta \qquad (2.40)$$

Compare the results with bi-axial and shear stresses conditions (Eq. 2.20).

- In a linear strain system, $\varepsilon_\theta = \varepsilon_x.\cos^2\theta$ or $\varepsilon_x\left(\dfrac{1+\cos 2\theta}{2}\right)$
- In a pure shear system and for $\theta = 45^\circ$, $\varepsilon_{45^\circ} = \varphi/2$. (Fig. 2.21$b$)

For maximum and minimum values of strains i.e. for Principal strains, differentiating with respect to θ and equating to zero,

$$\frac{d\varepsilon}{d\theta} = 0 - \frac{1}{2}(\varepsilon_x - \varepsilon_y)2\sin 2\theta + \varphi.\cos 2\theta = 0$$

or $$\frac{1}{2}(\varepsilon_x - \varepsilon_y)2\sin 2\theta = \varphi.\cos 2\theta$$

or $$\tan 2\theta = \frac{\varphi}{\varepsilon_x - \varepsilon_y} \qquad (2.41)$$

Values of principal strains can be obtained in a similar way as for principal stresses.

$$\text{Principal strain} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2} \qquad (2.42)$$

Mohr's Strain Circle

Comparing Eqs. 2.40 and 2.20, it may be observed that the Mohr's circle used for the stresses analysis can also be used for strain analysis. The linear strains can be taken along horizontal axis and shear strain along the vertical axis, the magnitude of the shear strain taken to be half. Thus in the strain circle (Fig. 2.23),

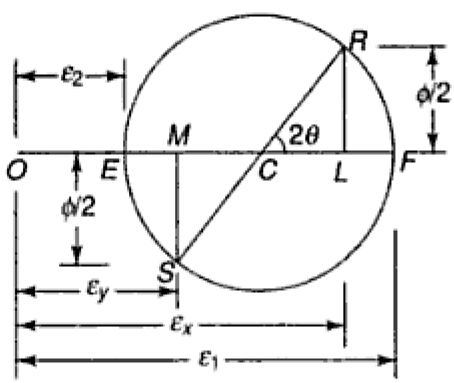


Fig. 2.23

$$OC = \frac{1}{2}(\varepsilon_x + \varepsilon_y)$$

And $$CR = \frac{1}{2}\sqrt{\{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2\}}$$

The construction of the strain circle, when three coplanar linear strains in three directions at a point are known, is exactly similar to that for the case when three coplanar stresses are known (Section 2.5).

Example 2.11 *Figure 2.24(a) shows the strains in three directions p, q and r in a plane, the magnitudes being 600×10^{-6}, -150×10^{-6} and 250×10^{-6}. Determine the magnitude and direction of the principal strains in this plane.*

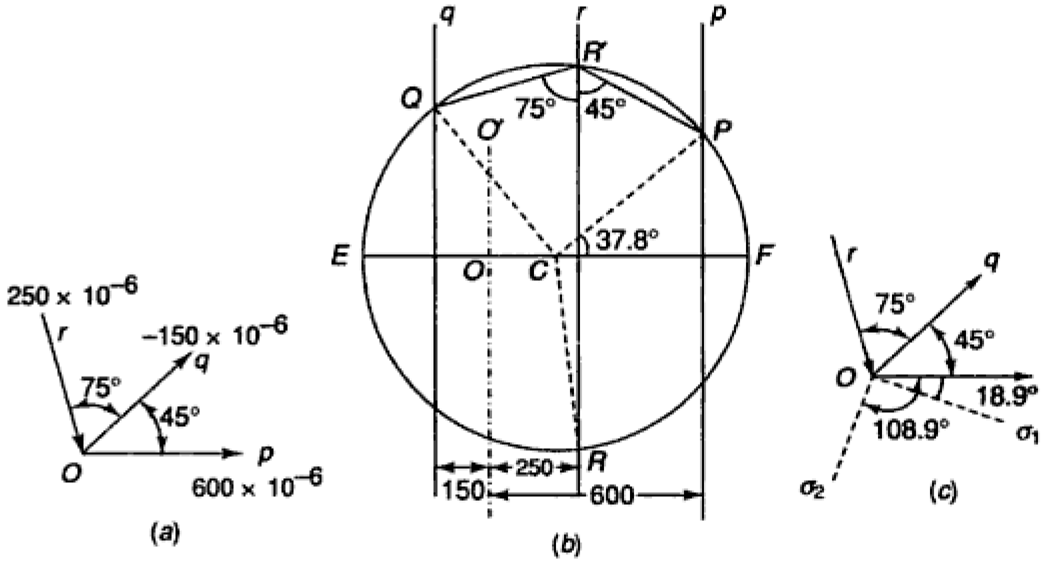


Fig. 2.24

Assuming no stress in a plane perpendicular to this plane, find the principal stresses at the point. Take E = 205 GPa and v = 0.3.

Solution

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta \quad \text{(Eq. 2.40)}$$

$$\therefore \quad \varepsilon_0 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) = 600 \times 10^{-6}$$

or $$\varepsilon_x = 600 \times 10^{-6} \quad \text{(i)}$$

$$\varepsilon_{45} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 90^\circ + \frac{1}{2}\varphi \sin 90^\circ$$

$$-150 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}\varphi$$

or $$\varepsilon_y + \varphi = -900 \times 10^{-6} \quad \text{(ii)}$$

$$\varepsilon_{120} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 240^\circ + \frac{1}{2}\varphi \sin 240^\circ$$

$$250 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) - \frac{1}{4}(600 \times 10^{-6} - \varepsilon_y) - \frac{\sqrt{3}}{4}\varphi$$

$$= \frac{1}{4}600 \times 10^{-6} + \frac{3}{4}\varepsilon_y - \frac{\sqrt{3}}{4}\varphi$$

$$\varepsilon_y - 0.577\,\varphi = 133.3 \times 10^{-6} \quad \text{(iii)}$$

Subtracting (iii) from (ii),

$$1.577\ \varphi = -1033 \times 10^{-6}$$
$$\varphi = -655 \times 10^{-6}$$
$$\varepsilon_y = -900 \times 10^{-6} - (-655 \times 10^{-6}) = -245 \times 10^{-6}$$

$$\tan 2\theta = \frac{\varphi}{\sigma_x - \sigma_y} = \frac{655 \times 10^{-6}}{(600 + 245)10^{-6}} = 0.775 \qquad \text{(Eq. 2.41)}$$

$$2\theta = 37.8^\circ \text{ or } \theta = 18.9^\circ \text{ and } 108.9^\circ$$

$$\text{The principal strains} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2} \qquad \text{(Eq. 2.42)}$$

$$= \frac{10^{-6}}{2}(600 - 245) \pm \frac{10^{-6}}{2}\sqrt{(600 + 245)^2 + 655^2}$$

$$= (177 \pm 535)10^{-6}$$

$$\varepsilon_1 = 712 \times 10^{-6} \quad \text{and} \quad \varepsilon_2 = -358 \times 10^{-6}$$

$$\sigma_1 = \frac{E(v\varepsilon_2 + \varepsilon_1)}{1 - v^2} = \frac{205\,000 \times 10^{-6}[0.3 \times (-358) + 712]}{1 - 0.3^2} = 136.2 \text{ MPa}$$

$$\sigma_2 = \frac{E(v\varepsilon_1 + \varepsilon_2)}{1 - v^2} = \frac{205\,000 \times 10^{-6}[0.3 \times 712 - 358]}{1 - 0.3^2} = -32.5 \text{ MPa}$$

Solution by Mohr's circle

Mohr's circle can be drawn as shown in Fig. 2.23(*b*) which is self-explanatory.
Major principal strain $\varepsilon_1 = OF = 712$ at 37.8/2 or 18.9° clockwise of plane *p*
Minor principal strain $\varepsilon_2 = OE = -358$ at 108.9° clockwise of plane *p*.
The results are shown in Fig. 2.23(*c*).

Summary

- Stresses on an inclined plane of system with direct and shear stresses are

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$
$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta$$
$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

- In complex systems of loading, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal. These are known as *principal planes* and the normal stresses across these planes as *principal stresses*.

- The inclination of principal planes is given by $\tan 2\theta = \dfrac{2\tau}{\sigma_x - \sigma_y}$
- The magnitude of major and minor principal stresses are given by

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

- Maximum shear stress $= \dfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} = \dfrac{1}{2}(\sigma_1 - \sigma_2)$
- A material acted upon by pure shear stresses on two perpendicular planes will have a tensile stress equal to the magnitude of the shear stress on the planes at 45° and a compressive stress of the same magnitude on the planes at 135° with no shear stress on these planes.
- Principal stresses from principal strains for 2-D,

$$\sigma_1 = \frac{E(\nu\varepsilon_2 + \varepsilon_1)}{1-\nu^2} \quad \text{and} \quad \sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1-\nu^2}$$

- Linear strain in an inclined plane,

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta$$

- Principal strains $= \dfrac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \dfrac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$

Review Questions

1. Show that in a direct stress system, the maximum shear stress in a body is half the magnitude of the applied stress.
2. Deduce expressions for stresses on an inclined plane in a body subjected to a bi-axial stress condition.
3. Show that shear stress in a body acted upon by two equal perpendicular stresses is zero.
4. Show that a body subjected to a pure shear is also acted upon by tensile and compressive stresses as well.
5. From first principles, show that a state of pure shear exists in a body subjected to equal perpendicular stresses of different nature.
6. What do you mean by principal planes and principal stresses? Derive the expression for principal stresses for a body subjected to direct and shear stresses.
7. What is Mohr's stress circle? How is it useful in the solution of stress analysis problems?
8. Deduce expression for the linear strain in a body in a direction inclined at angle θ with the x-axis when direct and shear strains along x- and y-directions are known.

9. A piece of material is subjected to two perpendicular tensile stresses of 300 MPa and 150 MPa. Determine the normal and shear stress components on a plane, the normal of which makes an angle of 40° with the 300 MPa stress. Also, find the resultant. (237.5 MPa; 74 MPa ; 248.8 MPa; 22.8°)

10. The stresses on two perpendicular planes through a point are 120 MPa tensile and 80 MPa compression along with 60 MPa shear. Determine the normal and shear stress components on a plane at 60° to that of the 120 MPa stress and also the resultant and its inclination with the normal component on the plane. (22 MPa; 116.5 MPa; 118.6 MPa; 79.3°)

11. Determine the position of the plane on which the resultant stress is most inclined to the normal in a system of two perpendicular compressive stresses of 120 MPa and 180 MPa. Also, find the value of the resultant stress. (11.5°; 147 MPa comp)

12. Solve Exercise 9 by Mohr's circle.

13. Draw Mohr's stress circle for a biaxial stress system having two direct stresses of 30 MPa (tensile) and 20 MPa (compressive). Determine the magnitude and the direction of the resultant stresses on planes which make angles of (i) 25°, and (ii) 70° with the 30-MPa stress. Also find the normal and shear stresses on these planes. (For 25° plane: 21 MPa (tensile), 19 MPa; 28.5 MPa ; 42°; For 70° plane: 14.2 MPa (comp.), 16 MPa; 21 MPa ; 131.5°)

14. At a point in steel bar the stresses on two mutually perpendicular planes are 10 MPa tensile and 5 MPa tensile whereas the shear stress across these planes is 2.5 MPa. Determine, using Mohr's circle, the normal as well as the shear stresses on a plane making an angle of 30° with the plane of the first stress. Also, find the magnitude and the direction of the resultant stress on the same plane. (10.9 MPa; 0.9 MPa; 10.95 MPa; 5°)

15. The normal stresses at a point in an elastic material are 100 MPa and 60 MPa respectively at right angle to each other with a shearing stress of 50 MPa. Determine the principal stresses and the position of principal planes if (i) both normal stresses are tensile, and (ii) 100 MPa stress is tensile and 60 MPa stress is compressive. Also determine the maximum shear stress and its plane in the two cases.
(133.8 MPa tensile, 26.2 MPa tensile, 34.1° and 124.1°, 53.8 MPa, 79.1°; 114.3 MPa tensile , 74.3 MPa comp., 16° and 106°, 94.3 MPa, 61°)

16. The resultant stress on a plane *BC* at a point in a material is 240 MPa tensile inclined at 30° to the normal to the plane as shown in Fig. 2.25. On a plane *AB* perpendicular to plane *BC*, the normal component of stress is 180 MPa. Determine the
(i) resultant stress on plane *AB*
(ii) principal stresses and principal planes
(ii) maximum shear stress.
(216.3 MPa; 56.5°; 314.6 MPa, 73 MPa; 41.7°, 131.7°; 120.8 MPa; 86.7°, 176.7°)

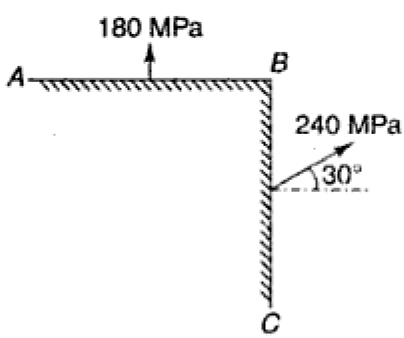


Fig. 2.25

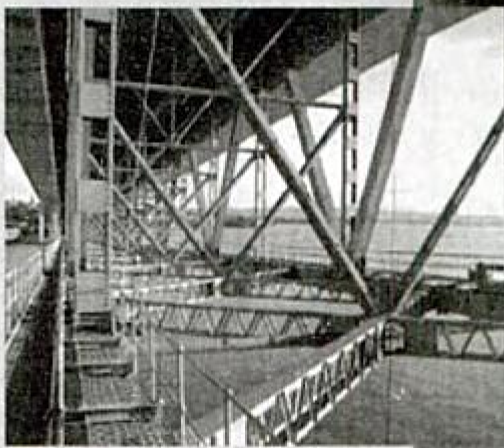
17. A piece of material is acted upon by tensile stresses of 50 MPa and 25 MPa at right angle to each other. Determine by ellipse of stress, the magnitude and direction of the resultant stress on a plane at 45° to the 50 MPa stress.
(39.5 MPa, 18° with normal stress, 27° with 100 MPa stress)

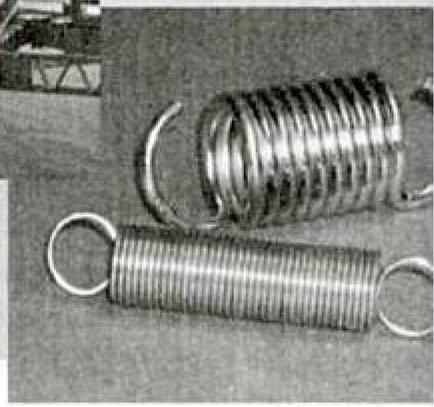
18. The stresses at a point in three coplanar directions are measured as σ_0 = 80 MPa (tensile), σ_{60} = 400 MPa (tensile) and σ_{120} = 200 MPa (compressive) where subscripts indicates the relative angular position of the planes in degrees. Determine the principal stresses and the planes.
[449 MPa (tensile) at 14° to 400 MPa and 251 MPa (compressive) at 16° to 200 MPa stress]

19. The readings of a strain gauge rosette inclined at 45° with each other are 4×10^{-6}, 3×10^{-6} and 1.6×10^{-6}, the first gauge being along x-axis. Determine the principal strains and the planes.
(4.04×10^{-6}, 1.58×10^{-6} ; 5° and 95°)

3

STRAIN ENERGY AND THEORIES OF FAILURES





3.1 INTRODUCTION

When an elastic body is loaded within elastic limits, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as *strain energy* and is denoted by U. It is recoverable without loss as soon as the load is removed from the body. However, if the elastic limit is exceeded, there is permanent set of deformations and the particles of the material of the body slide one over another. The work done in doing so is spent in overcoming the cohesion of the particles and the energy spent appears as heat in the strained material of the body. The concept of strain energy is very important in strength of materials as it is associated with the deformation of the body. The deflection of a body depends upon the manner of application of the load, i.e. whether the applied load is gradual, sudden or impact. If a body is acted upon by sudden or impact load, the instantaneous deformation is much more as compared to when the load is gradually applied. In such cases, strain energy is a convenient tool to solve problems associated with deflections.

A material is considered failed when a permanent or non-recoverable deformation occurs. There is either direct separation of particles as in case of brittle materials or slipping of particles as for ductile materials where plastic deformations also takes place. Machine components and structural members are generally designed on the hypothesis that the material will not yield during the expected loading conditions. For example, in a uniaxial loading, a machine component will be safe as long as the stress produced by the load is less than the yield stress of the material. In a biaxial loading system, it is not possible to predict directly by the above criterion. In such a case, it may require to find the principal stresses at any given point. Thus different criteria may be required to consider the safety of a component. The criteria used under various load conditions and type of materials are known as *theories of failure.*

3.2 STRAIN ENERGY

When a *gradual* or *static* load is applied to a body of an elastic material, the internal resistance or the stress increases linearly with the increase in deformation (Fig. 3.1) and therefore, the load-elongation or resistance-deformation diagram is a straight line within elastic limits. The maximum or the final resistance of the body is equal to the applied load. The work done in straining a material is equal to the area under the diagram at any instant or is the work done against the average resistance acting throughout.

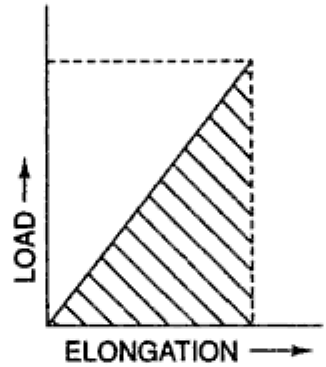


Fig. 3.1

Thus strain energy, U = Average resistance × Elongation = $\frac{1}{2}.P.\Delta$ (3.1)

The strain energy may also be expressed in the following forms:

$$U = \frac{1}{2}.(\sigma A).(\varepsilon.L) = \frac{1}{2}.\sigma.\varepsilon.AL = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume} \quad (3.2)$$

or $$U = \frac{1}{2}.\sigma.\frac{\sigma}{E}.\text{volume} = \frac{\sigma^2}{2E} \times \text{volume} \quad (3.3)$$

or $$U = \frac{P^2}{A^2}.\frac{1}{2E}.AL = \frac{P^2 L}{2AE} \quad (3.4)$$

Resilience

It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the strain energy per unit volume in simple tension or compression and is equal to $\sigma^2/2E$. It is also referred as *strain–energy density* and denoted by u.

Distinction between *strain energy* and *resilience* is hardly followed by authors and each is referred in place of the other.

Proof Resilience

It is the value of resilience at the elastic limit or at proof stress. It is also known as the *modulus of resilience* or *resilience modulus.*

Strain-energy density,

$$u = \frac{1}{2} \times \text{stress} \times \text{strain} \qquad \text{(From Eq. 3.2)}$$

$$= \frac{1}{2}.\sigma.\varepsilon = \frac{1}{2}.E\varepsilon.\varepsilon = \frac{E}{2}.\varepsilon^2 \qquad \text{(i)}$$

Now, consider an element of area of width $d\varepsilon$ located under the stress-strain diagram (Fig. 3.2).

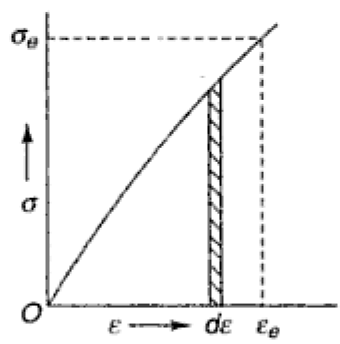


Fig. 3.2

Elemental Area = $\sigma.d\varepsilon = E\varepsilon.d\varepsilon$

Let ε_e denotes the strain corresponding the elastic limit of stress σ_e .

Then, area under the curve upto elastic limit = $E\int_0^{\varepsilon_e} \varepsilon.d\varepsilon = E\left(\frac{\varepsilon^2}{2}\right)_o^{\varepsilon_e} = \frac{E}{2}.\varepsilon_e^2$ which is same as (i) above. Thus proof resilience is the area under the stress-strain diagram upto elastic limit.

Modulus of Toughness

It is the strain energy per unit volume required to cause the material to rupture. It is the area under the stress-strain diagram upto rupture point (Fig. 3.3). Thus it is a measure of the ability of a material to absorb energy before fracture.

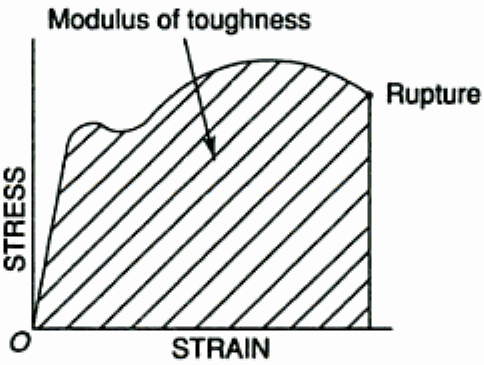


Fig. 3.3

3.3 STRAIN ENERGY (THREE-DIMENSIONAL STRESS SYSTEM)

Consider a unit cube acted upon by three principal stresses σ_1, σ_2 and σ_3 (Fig. 3.4). Let the corresponding strains be ε_1, ε_2 and ε_3. Then for gradually applied stresses,

Strain energy,

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

$$= \frac{1}{2}\sigma_1\left(\frac{\sigma_1 - v\sigma_2 - v\sigma_3}{E}\right) + \frac{1}{2}\sigma_2\left(\frac{\sigma_2 - v\sigma_3 - v\sigma_1}{E}\right) + \frac{1}{2}\sigma_3\left(\frac{\sigma_3 - v\sigma_1 - v\sigma_2}{E}\right)$$

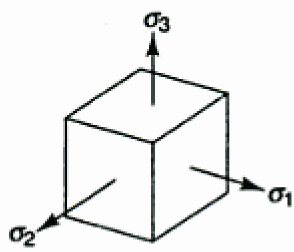


Fig. 3.4

$$= \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \text{ per unit volume} \qquad (3.5)$$

- For a system with equal principal stresses $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$,

$$U = \frac{1}{2E}(\sigma^2 + \sigma^2 + \sigma^2 - 2v(\sigma\sigma + \sigma\sigma + \sigma\sigma) = \frac{3\sigma^2}{2E}(1-2v) \qquad (3.6)$$

As $E = 3K(1 - 2v)$

$$\text{Thus, } U = \frac{3\sigma^2}{2\times 3K(1-2v)}(1-2v) = \frac{\sigma^2}{2K} \qquad (3.7)$$

- For a two-dimensional system

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2v\sigma_1\sigma_2) \text{ per unit volume} \qquad (3.8)$$

3.4 SHEAR STRAIN ENERGY

Consider a block with dimensions *L*, *b* and *h* as shown in Fig. 3.5. Assume it to be rigidly fixed to the ground. A shear force *P* is applied gradually along the top surface.

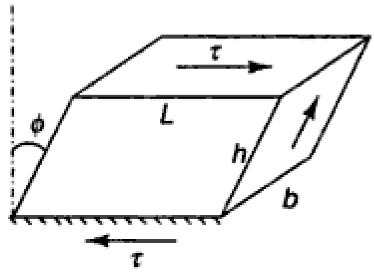


Fig. 3.5

Strain energy, U = Work done in straining = $\frac{1}{2}$ × Final couple × Angle turned

$$= \frac{1}{2} \times \text{Final force} \times h \times \varphi$$

$$= \frac{1}{2} \times (\text{Shear stress} \times \text{Area}) \times h \times \varphi$$

$$= \frac{1}{2}.\tau.(L.b).h.\frac{\tau}{G} \qquad \text{....... (As } G = \tau/\varphi \text{ or } \varphi = \tau/G)$$

$$= \frac{\tau^2}{2G}.(L.b.h)$$

$$= \frac{\tau^2}{2G} \times \text{Volume} \qquad (3.9)$$

or Shear strain energy per unit volume = $\frac{\tau^2}{2G}$ (3.10)

It is similar to $\sigma^2/2E$ for direct stress.

3.5 SHEAR STRAIN ENERGY (THREE-DIMENSIONAL STRESS SYSTEM)

Consider a unit cube acted upon by three principal stresses σ_1, σ_2 and σ_3 as before. The total work done by the external forces cause

- change of volume due to application of direct stresses and
- distortion due to shearing stresses which do not affect the volumetric change.

Thus,

Total strain energy = Volumetric strain energy + Shear strain energy

Now total strain energy,

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \text{ per unit volume} \qquad \text{(Eq.3.5)}$$

Volumetric strain energy,

$$U_v = \frac{1}{2}\text{Average stress} \times \text{Volumetric strain}$$

$$= \frac{1}{2}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right) \times \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E} \qquad \text{(Using Eq. 1.23)}$$

$$= \frac{1}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 (1 - 2\nu) \qquad (3.11)$$

Thus, shear strain energy

$$= \frac{1}{2E}(\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)-\frac{1}{6E}(\sigma_1+\sigma_2+\sigma_3)^2(1-2\nu)$$

$$= \frac{1}{6E}\{3(\sigma_1^2+\sigma_2^2+\sigma_3^2)-6\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\} -$$

$$\frac{1}{6E}\{(\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\}(1-2\nu)$$

$$= \frac{1}{6E}[3(\sigma_1^2+\sigma_2^2+\sigma_3^2)-6\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)-(\sigma_1^2+\sigma_2^2+\sigma_3^2)-$$

$$2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)+2\nu(\sigma_1^2+\sigma_2^2+\sigma_3^2)+4(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]$$

$$= \frac{1}{6E}[(\sigma_1^2+\sigma_2^2+\sigma_3^2)(3-1+2\nu)-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)(6\nu+2-4\nu)]$$

$$= \frac{1+\nu}{6E}[2(\sigma_1^2+\sigma_2^2+\sigma_3^2)-2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]$$

$$= \frac{1+\nu}{6\times 2G(1+\nu)}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

$$= \frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \qquad (3.12)$$

- For a two-dimensional stress system, the relation for shear strain energy reduces to $\frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2)^2+(-\sigma_1)^2]=\frac{1}{6G}(\sigma_1{}^2+\sigma_2{}^2-\sigma_1\sigma_2)$ (3.13)

Example 3.1 *Compare the strain energies of the bars shown in Fig. 3.6 (ii), (iii), (iv) and (v) with the strain energy of bar (i) for a constant load P in all cases. Smaller cross-sectional areas are half of the larger cross-sectional areas.*

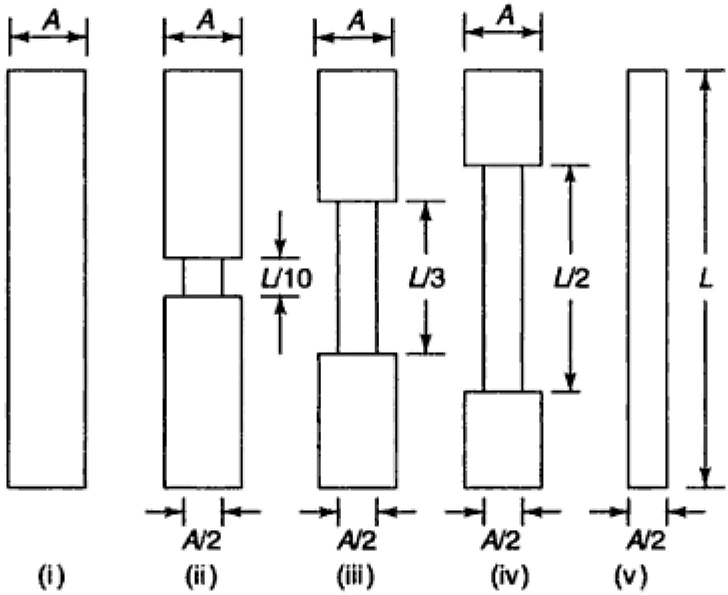


Fig. 3.6

Solution Let A = area of larger cross-section
a = area of smaller cross-section
L = Total length of bar
L_A = length of section with larger area
L_a = length of section with smaller area

Thus

As the load is the same in all cases,
Stress in the larger section, $\sigma = P/A$

Stress in the smaller section, $\sigma_1 = \dfrac{P}{a} = \dfrac{P/A}{a/A} = \dfrac{\sigma}{1/2} = 2\sigma$

(i) Strain energy of bar (i), $U_1 = \dfrac{\sigma^2}{2E}.AL$

(ii) Strain energy of non-uniform section of bar of Fig. 3.6(ii),

U = Strain energy of larger section + strain energy of smaller section

$$= \frac{\sigma^2}{2E}.A.L_A + \frac{(2\sigma)^2}{2E}.a.L_a$$

$$\text{Ratio} = \frac{\dfrac{\sigma^2}{2E}.A.L_A + \dfrac{(2\sigma)^2}{2E}.a.L_a}{\dfrac{\sigma^2}{2E}.AL} = \frac{2aL_A + 4a.L_a}{2a.L} = \frac{L_A + 2L_a}{L}$$

Thus, $U/U_1 = 0.9 + 2 \times 0.1 = 1.1$

Similarly,

(iii) $U/U_1 = 2/3 + 2 \times 1/3 = 1.33$ (iv) $U/U_1 = 0.5 + 2 \times 0.5 = 1.5$

(v) $U/U_1 = 0 + 2 \times 1 = 2$

Example 3.2 *Compare the strain energies of the bars shown in Fig. 3.6 (ii), (iii), (iv) and (v) with the strain energy of bar (i), when the bars are subjected to maximum permissible stress.*

Solution In this case, the maximum value of stress in all the cases is σ which means that in the first case the maximum stress is uniform, whereas in other cases the maximum stress is in the sections with smaller cross-sections.

(i) *Bar of Fig. 3.6 (i):*

Stress in the uniform section is σ.

Strain energy, $U_1 = \dfrac{\sigma^2}{2E} \times AL$

(ii) *Bar of Fig. 3.6 (ii):*

Stress in the smaller section = maximum permissible stress = σ

Stress in the larger section = $\sigma.\dfrac{a}{A} = \dfrac{\sigma}{2}$

Strain energy of non-uniform section of Fig. 3.6(ii),

U = Strain energy of larger section + strain energy of smaller section

$$= \frac{(\sigma/2)^2}{2E}.A.L_A + \frac{\sigma^2}{2E}.a.L_a$$

$$\text{Ratio} = \frac{\frac{(\sigma/2)^2}{2E}.AL_A + \frac{\sigma^2}{2E}.a.L_a}{\frac{\sigma^2}{2E}.AL} = \frac{\frac{2aL_A}{4} + a.L_a}{2aL}$$

$$= \frac{1}{4L}(L_A + 2L_a) = \frac{1}{4}(0.9 + 2\times.1) = 0.275$$

Similarly for other bars,

(iii) $\frac{U}{U_1} = \frac{1}{4}[0.667 + 2 \times 0.333] = 0.333$

(iv) $\frac{U}{U_1} = \frac{1}{4}[0.5 + 2\times 0.5] = 0.375$ (v) $\frac{U}{U_1} = \frac{1}{4}[0 + 2\times 1] = 0.5$

Example 3.3 *Compare the strain energies per unit volume of the bars shown in Fig. 3.6 (ii), (iii), (iv) and (v) with the strain energy per unit volume of bar (i) when the bars are subjected to maximum permissible stress.*

Solution

(i) Stress in the uniform section = maximum permissible stress = σ

Strain energy per unit volume, $U_1 = \frac{\sigma^2}{2E}$

(ii) For non-uniform section of Fig. 3.6(ii),

Stress in the smaller section = maximum permissible stress = σ

Stress in the larger section = $\sigma.\frac{a}{A} = \frac{\sigma}{2}$

Volume = $A.L_A + aL_a$

Strain energy of non-uniform section of Fig. 3.6(ii)

$$= \frac{(\sigma/2)^2}{2E}.A.L_A + \frac{\sigma^2}{2E}.a.L_a$$

Strain energy per unit volume of non-uniform section,

$$\frac{\frac{(\sigma/2)^2}{2E}.A.L_A + \frac{\sigma^2}{2E}.a.L_a}{A.L_A + aL_a} = \frac{\sigma^2}{2E}\left(\frac{\frac{1}{4}.2a.L_A + a.L_a}{2a.L_A + aL_a}\right) = \frac{\sigma^2}{2E}\left(\frac{L_A/2 + L_a}{2L_A + L_a}\right)$$

$$\text{Ratio, } \frac{U}{U_1} = \frac{\frac{\sigma^2}{2E}\left(\frac{L_A/2 + L_a}{2L_A + L_a}\right)}{\frac{\sigma^2}{2E}} = \frac{(L_A/2 + L_a)/L}{(2L_A + L_a)/L} = \frac{0.9/2 + 0.1}{2\times 0.9 + 0.1} = 0.289$$

Similarly,

(iii) $\dfrac{U}{U_1} = \dfrac{0.6667/2 + 0.3333}{2 \times 0.6667 + 0.3333} = 0.4$ (iv) $\dfrac{U}{U_1} = \dfrac{0.5/2 + 0.5}{2 \times 0.5 + 0.5} = 0.5$

(v) $\dfrac{U}{U_1} = \dfrac{0/2 + 1}{2 \times 0 + 1} = 1$

Example 3.4 *The cross-sections of two bars A and B made up of the same material and each 320-mm long are as follows:*

- *Bar A: 24-mm diameter for a length of 80 mm and 48 mm for the remaining 240 mm*
- *Bar B: 24-mm diameter for a length of 240 mm and 48 mm for the remaining 80 mm*

An axial blow to bar A produces a maximum instantaneous stress of 160 MPa. Determine the

(i) maximum instantaneous stress produced by the same blow to bar B.
(ii) ratio of energies stored by the two bars when subjected to maximum permissible stress.
(iii) ratio of energies per unit volume of the two bars when subjected to maximum permissible stress.

Solution (Refer Fig. 3.7.)

For the same blow to bar *B*, the strain energy produced by the blow should equal to that produced by the blow to the first bar.

In bar A,

Maximum instantaneous stress in the smaller cross-section = 160 MPa

Maximum instantaneous stress in the larger cross-section = $160 \times \left(\dfrac{24}{48}\right)^2 = 40$ MPa

In bar B,

Let maximum instantaneous stress in the smaller cross-section = σ

Then maximum instantaneous stress in the larger cross-section = $\sigma/4$

Strain energy of bar *A* = strain energy of bar *B*

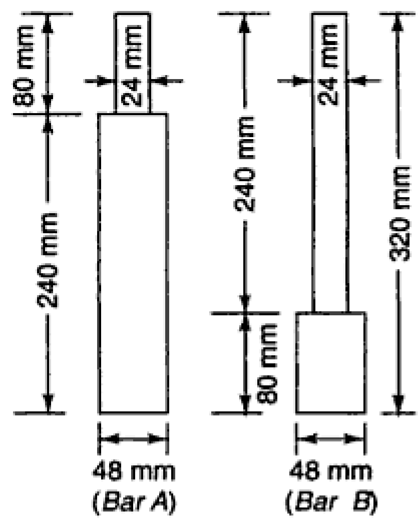


Fig. 3.7

$$\frac{40^2}{2E} \times \frac{\pi}{4}(48)^2 \times 240 + \frac{160^2}{2E} \times \frac{\pi}{4}(24)^2 \times 80$$
$$= \frac{(\sigma/4)^2}{2E} \times \frac{\pi}{4}(48)^2 \times 80 + \frac{\sigma^2}{2E} \times \frac{\pi}{4}(24)^2 \times 240$$

Dividing throughout by $\dfrac{\pi}{4} \times 24^2 \times \dfrac{80}{2E}$,

$$40^2(12+16) = \sigma^2\left(\frac{1}{4}+3\right) \text{ or } \sigma = 117.4 \text{ MPa}$$

(ii) Let maximum stress in the smaller cross-section = σ
Then stress in the larger cross-section = $\sigma/4$
Ratio of strain energies

$$\frac{U_B}{U_A} = \frac{\dfrac{(\sigma/4)^2}{2E}\times\dfrac{\pi}{4}(48)^2\times 80+\dfrac{\sigma^2}{2E}\times\dfrac{\pi}{4}(24)^2\times 240}{\dfrac{(\sigma/4)^2}{2E}\times\dfrac{\pi}{4}(48)^2\times 240+\dfrac{\sigma^2}{2E}\times\dfrac{\pi}{4}(24)^2\times 80}$$

Dividing throughout by $\dfrac{\sigma^2}{2E}\times\dfrac{\pi}{4}(24)^2\times 80$

$$\frac{U_B}{U_A} = \frac{(1/16)\times 2^2+3}{(1/16)2^2\times 3+1} = 1.857$$

(iii) The ratio of strain energies per unit volume of two bars for the same maximum stress in the smaller cross-sections,

Volume of bar $A = \dfrac{\pi}{4}(48)^2\times 240+\dfrac{\pi}{4}(24)^2\times 80\ \text{mm}^3$;

Volume of bar $B = \dfrac{\pi}{4}(48)^2\times 80+\dfrac{\pi}{4}(24)^2\times 240\ \text{mm}^3$;

$$\frac{U_B}{U_A} = \frac{\left[\dfrac{(\sigma/4)^2}{2E}\times\dfrac{\pi}{4}(48)^2\times 80+\dfrac{\sigma^2}{2E}\times\dfrac{\pi}{4}(24)^2\times 240\right]\dfrac{4/\pi}{48^2\times 80+24^2\times 240}}{\left[\dfrac{(\sigma/4)^2}{2E}\times\dfrac{\pi}{4}(48)^2\times 240+\dfrac{\sigma^2}{2E}\times\dfrac{\pi}{4}(24)^2\times 80\right]\dfrac{4/\pi}{48^2\times 240+24^2\times 80}}$$

$$= 1.857\times\frac{2^2\times 3+1}{2^2+3} = 1.857\times 1.857 = 3.448$$

Example 3.5 *A steel rope lowers a load of 9.5 kN with a uniform velocity of 750 mm/s. When the length of the rope unwound is 8 m, it suddenly gets jammed and the load is brought to a halt. Determine the stress developed in the rope due to sudden stoppage and the maximum instantaneous elongation if the diameter of the rope is 20 mm. E for steel = 205 GPa.*

Solution $A = \dfrac{\pi}{4}.(20)^2 = 100\pi\ \text{mm}^2$

K.E. of the load $= \dfrac{Wv^2}{2g} = \dfrac{9500\times 750^2}{2\times 9810} = 272\ 362$ N.m

Strain energy gained by the rope $= \dfrac{\sigma^2}{2E}AL = \dfrac{\sigma^2}{2\times 205\ 000}\times 100\pi\times 8000$

$= 6.13\ \sigma^2$ N.m

When the rope is suddenly gets jammed, its kinetic energy is converted into strain energy.

Thus equating the two, 6.13 $\sigma^2 = 272\ 362$ or $\sigma = 210.8$ MPa

$$\text{Maximum instantaneous elongation} = \frac{\sigma L}{E} = \frac{210.8 \times 8000}{205\ 000} = 8.226 \text{ mm}$$

3.6 STRESSES DUE TO VARIOUS TYPES OF LOADING

A loading may be gradual, sudden or by impact (shock). The case of gradual loading has already been discussed in Section 3.2.

Suddenly Applied Load

When a load is applied suddenly, the load W is the same throughout whereas the internal resistance set up in the body increases linearly with the deformation, i.e. it is zero in the beginning and maximum (equal to applied load) when the deformation is Δ.

Strain energy stored in the material = Average resistance × Elongation = $\frac{1}{2}.(\sigma A).\Delta$

Work done by the load = $W.\Delta$

$$\text{Equating the two, } \frac{1}{2}.(\sigma A).\Delta = W.\Delta \text{ or } \sigma = \frac{2W}{A} \tag{3.14}$$

i.e. the maximum stress induced in the body is twice the stress induced by the load of the same magnitude applied gradually.

Impact or Shock Load

Let a load W drop through a height h before it commences to deform the body. After falling on the collar of the body as shown in Fig. 3.8, a few oscillations of the collar take place and finally the load takes up the same position as is taken by gradually applied load if the limit of proportionality is not exceeded. Of course, the instantaneous extension and the stress are much greater than the steady state values.

Work done (potential energy loss) by the load = Strain energy stored in the material

$$W(h + \Delta) = \frac{\sigma^2}{2E} \times AL \qquad \text{(Eq. 3.3)}$$

where σ is the stress due to a gradually applied load P that causes the same deflection Δ.

or
$$W\left(h + \frac{\sigma L}{E}\right) = \frac{\sigma^2}{2E} \times AL$$

Multiplying throughout by $2E/AL$ and rearranging,

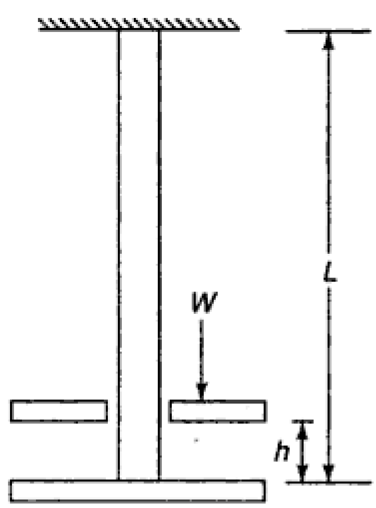


Fig. 3.8

$$\sigma^2 - \left(\frac{2W}{A}\right)\sigma - \frac{2WEh}{AL} = 0$$

This is a quadratic in σ and the solution is given by,

$$\sigma = \frac{W}{A} \pm \frac{1}{2}\sqrt{\left(\frac{2W}{A}\right)^2 + 4 \times \frac{2WEh}{AL}} = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right) \qquad (3.15)$$

If $h = 0$, $\sigma = 2W/A$, i.e. the case of a suddenly applied load.

Example 3.6 *A 3.2-m long bar of 16 mm in diameter hangs vertically and has a collar attached at the lower end. Determine the maximum stress induced when a weight of 80 kg falls from a height of 32 mm on the collar.*

If the bar is turned down to half the diameter along half of its length, what will be the value of the maximum stress? E =205 GPa.

Solution $A = (\pi/4).16^2 = 64\,\pi$; $h = 32$ mm; $W = 80 \times 9.81 = 784.8$ N; $L = 3.2$ m

$$\sigma = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right) = \frac{784.8}{64\pi}\left(1 + \sqrt{1 + \frac{2 \times 64\pi \times 205\,000 \times 32}{784.8 \times 3200}}\right)$$

$$= 3.903(1 + 32.4) = 130.4 \text{ MPa}$$

Note that with only a 32-mm drop the maximum stress is 32.4 times more than that for static stress.

Now, diameter of the reduced section = 16/2 = 8 mm

Area of the reduced section = $(\pi/4).8^2 = 16\,\pi$

When the bar is turned down to half the diameter along half of its length, let P be the equivalent load to induce the same maximum stress.

$$\therefore\ \Delta = \frac{P \times 1600}{16\pi \times 205\,000} + \frac{P \times 1600}{64\pi \times 205\,000} = 0.000\,194\,P = kP$$

...(Taking $k = 0.000\,194$)

Using the energy equation

$$W(h + \Delta) = \frac{1}{2}P.\Delta$$

$$W(h + kP) = \frac{1}{2}P.\,kP$$

$$P^2 - 2WP - 2Wh/k = 0 \qquad \text{(multiplying throughout by } 2/k)$$

Solving,

$$P = \frac{2W + \sqrt{(-2W)^2 - 4(-2Wh/k)}}{2} = 784.8 + \sqrt{784.8^2 + \frac{2 \times 784.8 \times 32}{0.000\,194}} = 16\,894 \text{ N}$$

The maximum stress (in the smaller section) = $16\,894/16\pi = 336$ MPa

The maximum extension = $kp = 0.000\,194 \times 16\,894 = 3.28$ mm

Example 3.7 *A lift is operated by three ropes each having 28 wires of 1.4 mm diameter. The cage weighs 1.2 kN and the weight of the rope is 4.2 N/m length. Determine the maximum load carried by the lift if each wire is of 36 m length and the lift operates (i) without any drop (ii) with a drop of 96 mm during operations.*

E (rope) = 72 GPa and allowable stress = 115 MPa

Solution Total area of cross section, $A = \frac{\pi}{4}.(1.4)^2 \times 3 \times 28 = 129.3\ \text{mm}^2$

The maximum stress occurs at the top of the wire rope where the weight of the rope is maximum.

Thus maximum load = weight of cage + weight of rope

$= 1200 + 3 \times 36 \times 4.2 = 1653.6$ N

Initial stress in the rope, $\sigma = \frac{1653.6}{129.3} = 12.8$ MPa

Equivalent static stress available for carrying the load = 115 – 12.8 = 102.2 MPa

Thus, equivalent static load that can be carried,

$P_e = 102.2 \times 129.3 = 13\,214$ N

The extension of the rope, $\Delta = \frac{102.2 \times 36\,000}{72\,000} = 51.1$ mm

(i) With no drop, Let W be the weight which can be applied suddenly, $W.\Delta = \frac{1}{2} P.\Delta$

or $W = 13214/2 = 6607$ N or 6.607 kN

(ii) With 96 mm drop, Let W be the weight,

$$W(h + \Delta) = \frac{1}{2} P.\Delta \quad \text{or} \quad W(96 + 51.1) = \frac{1}{2} \times 13214 \times 51.1$$

or $W = 2295$ N or 2.295 kN

Example 3.8 *A vertical composite tie bar rigidly fixed at the upper end consists of a steel rod of 16-mm diameter enclosed in a brass tube of 16-mm internal diameter and 24-mm external diameter, each being 2 m long. Both are fixed together at the ends. The tie bar is suddenly loaded by a weight of 8 kN falling through a distance of 4 mm. Determine the maximum stresses in the steel rod and the brass tube.*

E_s = 205 GPa and E_b = 100 GPa

Solution Refer Fig. 3.9.

$A_s = (\pi/4)16^2 = 64\,\pi$, $\quad A_b = (\pi/4)(24^2 - 16^2) = 80\,\pi$

Let x = Extension of bar in mm

$$\sigma_s = \frac{E_s.x}{L} \text{ and } \sigma_b = \frac{E_b.x}{L}$$

$$\text{Strain energy of the bar} = \frac{\sigma_s^2}{2E_s} A_s L + \frac{\sigma_b^2}{2E_b} A_b L$$

$$= \frac{E_s^2 x^2}{L^2.2E_s} A_s L + \frac{E_b^2 x^2}{L^2.2E_b} A_b L$$

$$= \frac{E_s x^2}{2L} A_s + \frac{E_b x^2}{2L} A_b$$

$$= \frac{x^2}{2L}(E_s A_s + E_b A_b)$$

$$= \frac{x^2}{2 \times 2000}(205\ 000 \times 64\pi + 100\ 000 \times 80\pi)$$

$= 16\ 588\ x^2$ N.mm

Potential energy lost by the weight

$= W(h + x) = 8000\ (4 + x)$ N.mm

Equating the two, $16\ 588\ x^2 = 8000\ (4 + x)$

or $x^2 - 0.4823x - 1.9292 = 0$

or $$x = \frac{1}{2}(0.4823 + \sqrt{0.4823^2 + 4 \times 1.9292})$$

$= 1.6508$ mm

$$\sigma_s = \frac{E_s . x}{L} = \frac{205\ 000 \times 1.6505}{2000} = 169.2 \text{ MPa}$$

$$\sigma_b = \frac{E_b . x}{L} = \frac{100\ 000 \times 1.6505}{2000} = 82.54 \text{ MPa}$$

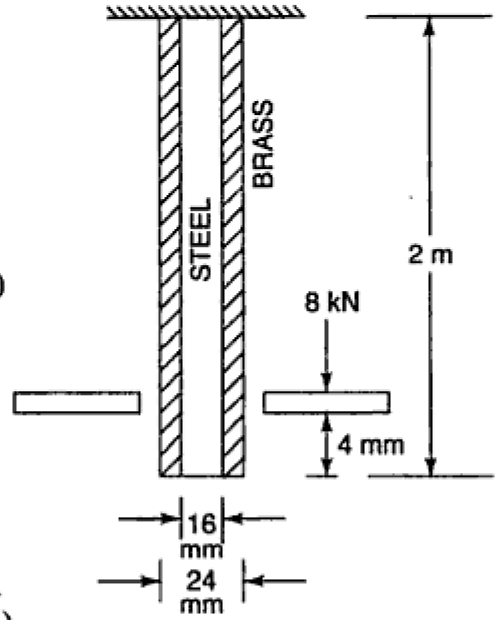


Fig. 3.9

3.7 THEORIES OF FAILURES

As mentioned in the introduction, in simple systems with only one kind of stress, it is easy to anticipate the failure, but in complex stress systems in which direct as well as shear stresses act, it is not easy to do so.

The main theories of failure are discussed below. σ_1, σ_2 and σ_3 denote the principal stresses in any complex system and σ the tensile stress at the elastic limit in simple tension.

(i) Maximum Principal Stress Theory (Rankine's Theory)

According to this theory the failure of material will occur when the maximum principal stress in the complex stress system attains the value of the maximum stress at the elastic limit in simple tension. Thus this theory states that the maximum principal stress must not exceed the working stress for the material.

Let σ_x, σ_x, τ = Direct and shear stresses on given planes in the complex system

and σ_1 = Maximum principal stress

Then $$\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$$

It should not be more than maximum stress in simple tension σ,

or in the limit, $\sigma_1 = \sigma$

This theory is found to give good results when applied to brittle materials such as cast iron.

(ii) Maximum Shear Stress Theory (Guest's and Tresca's Theory)

This theory states that the failure occurs when the maximum shear stress in the complex system attains the value of the maximum shear stress at the elastic limit in simple tension.

The maximum value of shear stress in terms of principal stresses σ_1 and σ_2,

$$\tau = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$$

In a simple direct stress system, the maximum shear stress = $\sigma/2$

Thus $(\sigma_1 - \sigma_2)/2$ must not be more than $\sigma/2$,

or in the limit, $\sigma_1 - \sigma_2 = \sigma$

However, this applies to values of principal stresses of opposite type, i.e. if one is tensile, the other is compressive. When the principal stresses are alike, either both are tensile or compressive, then according to the above relation, if σ_1 is the maximum principal stress, it will be more than the limiting value of stress σ which is not possible. In that case, the higher value of principal stress must be less than the limiting value σ.

In the limit, $\sigma_1 = \sigma$ or $-\sigma$

This theory is preferred in case of ductile materials such as mild steel.

(iii) Maximum Principal Strain Theory (St. Venant's Theory)

According to this theory, the maximum principal strain in the complex stress system must be less than the elastic limit in simple tension if there is to be no failure.

In the limit,

$$\varepsilon_1 = \left(\frac{\sigma_1 - \nu\sigma_2 - \nu\sigma_3}{E}\right) = \frac{\sigma}{E} \quad \text{or} \quad \sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma$$

This theory is not used in general as it is found to give satisfactory results in particular cases only.

(iv) Maximum Strain Energy Theory (Haigh's Theory)

This theory is based on the principle that the work done in bringing a body to a particular state is independent of the method applied to bring the body to that state. According to this theory, the failure takes place when the strain energy per unit volume of a body reaches the value of strain energy at elastic limit in simple tension.

In the limit,

$$\frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)) = \frac{\sigma^2}{2E} \qquad \text{(Eq. 3.5)}$$

or $\quad \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$

This theory is found satisfactory for ductile materials.

(v) Maximum Shear Strain Energy Theory (Mises' and Henkey's Theory)

This theory states that the failure takes place when the shear strain energy in a complex system becomes equal to that in simple tension.

Shear strain energy in a complex system

$$= \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \qquad \text{(Eq. 3.12)}$$

Shear strain energy in simple tension is found by inserting $\sigma_1 = \sigma$, $\sigma_2 = 0$ and $\sigma_3 = 0$ in the above expression, i.e.

Shear strain energy in simple tension

$$= \frac{1}{12G}[(\sigma - 0)^2 + (0 - 0)^2 + (0 - \sigma)^2] = \frac{2\sigma^2}{12G} = \frac{\sigma^2}{6G}$$

Therefore, in the limit

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 2\sigma^2$$

For a two-dimensional stress system, the above relation may be reduced to

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2)^2 + (-\sigma_1)^2] = 2\sigma^2$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma^2$$

This theory gives good results when applied to ductile materials.

As shear stress and shear strain energy theories depend upon the stress differences, a material has no chance of failure if the principal stresses are of the same nature and magnitude since the difference will be negligible. Thus these theories should not be applied in such cases.

Example 3.9 *Principal stresses at a point in an elastic material are 100 MPa tensile, 50 MPa tensile and 25 MPa compressive. Determine the factor of safety against failure based on various theories. The elastic limit in simple tension is 220 MPa and Poisson's ratio 0.3.*

Solution

(i) *Maximum principal stress theory*

Failure takes place when the maximum principal stress reaches the value of maximum stress at the elastic limit.

Thus maximum principal stress, $\sigma = 100$ MPa

Factor of safety = 220/100 = 2.2

(ii) *Maximum shear stress theory*

$\sigma = 100 - (-25) = 125$ MPa

Factor of safety = 220/125= 1.76

(iii) *Maximum principal strain theory*

$$\sigma_1 - v\sigma_2 - v\sigma_3 = 100 - 0.3 \times 50 - 0.3 \times (-25) = 92.5 \text{ MPa}$$

Factor of safety = 220/92.5 = 2.37

(iv) *Maximum strain energy theory*

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$= 100^2 + 50^2 + (-25)^2 - 2 \times 0.3 \times [100 \times 50 + 50 \times (-25) + (-25) \times 100]$

$= 13\,125$

$\sigma = 114.6$ MPa

Factor of safety = 220/114.6 = 1.92

(v) *Maximum shear strain energy theory*

$$2\sigma^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$= (100 - 50)^2 + (50 + 25)^2 + (-25 - 100)^2 = 23\,750$$

$\sigma^2 = 11\,875$ $\quad\quad$ $\sigma = 108.97$

Factor of safety = 220/108.97= 2.02

Example 3.10 *A bolt is acted upon by an axial pull of 16 kN alongwith a transverse shear force of 10 kN. Determine the diameter of the bolt required according to different theories. Elastic limit of the bolt material is 250 MPa and a factor of safety 2.5 is to be taken. Poisson's ratio is 0.3.*

Solution The permissible stress in simple tension = 250/2.5 = 100 MPa

Let the required area of cross-section and the diameter of the bolt be a and d respectively under different theories.

The applied tensile stress = 16 000/a

The applied shear stress = 10 000/a

Maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$

$$= \frac{1}{2a}(16\,000) + \frac{1}{2a}\sqrt{16\,000^2 + 4 \times 10\,000^2} \quad ...(\sigma_y = 0)$$

$= (8000 + 12\,806)/a = 20\,806/a$ (tensile)

Minimum principal stress, $\sigma_2 = (8000 - 12\,806)/a = 4806/a$ (compressive)

(i) *Maximum principal stress theory*:

Maximum principal stress, $\sigma_1 = 20\,806/a$

Thus $\quad$ $20\,806/a = 100$

$$\frac{\pi}{4}d^2 = 208.06 \text{ or } d = 16.28 \text{ mm}$$

(ii) *Maximum shear stress theory*

Maximum shear stress = $[20\,806 - (-4806)]/2a = 12\,806/a$

Maximum shear stress in simple tension = 100/2 = 50 MPa

$\therefore$ $\quad$ $12\,806/a = 50$

$$\frac{\pi}{4}d^2 = 256.12 \text{ or } d = 18.05 \text{ mm}$$

(iii) *Maximum principal strain theory*

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = [20\,806 - 0.3 \times (-4806)]/a = 22\,247.8/a \quad ...(\sigma_3 = 0)$$

$22\,247.8/a = 222.48$

$$\frac{\pi}{4}d^2 = 222.48 \text{ or } d = 16.83 \text{ mm}$$

(iv) *Maximum strain energy theory*

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$$

$$[20\,806^2 + (-4806)^2 - 2 \times 0.3 \times 20\,806 \times (-4806)]/a = 100^2$$

$$a = 227.16 \text{ mm}^2$$

$$\frac{\pi}{4} d^2 = 227.16 \text{ or } d = 17.0 \text{ mm}$$

(v) *Maximum shear strain energy theory*

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 2\sigma^2$$

$$[(20\,806 + 4806)^2 + (-4806)^2 + (-20\,806)^2]/a^2 = 2 \times 100^2$$

$$a^2 = 55\,598 \text{ or } a = 235.8 \text{ mm}^2$$

or $$\frac{\pi}{4} d^2 = 235.8 \text{ or } d = 17.32 \text{ mm}$$

3.8 GRAPHICAL REPRESENTATION OF THEORIES OF FAILURES

In a two-dimensional stress system, the limits of principal stresses according to different theories can be shown graphically as under:

In a two-dimensional system, σ_3 is taken to be zero and the values of principal stresses σ_1 and σ_2 are taken along *x*- and *y*-axes respectively. Positive values of σ_1 are taken towards right of the *y*-axis and negative towards left. Similarly, positive values of σ_2 are taken upwards and negative downwards of the *x*-axis. The elastic limit σ may be taken to be the same both in tension as well as in compression.

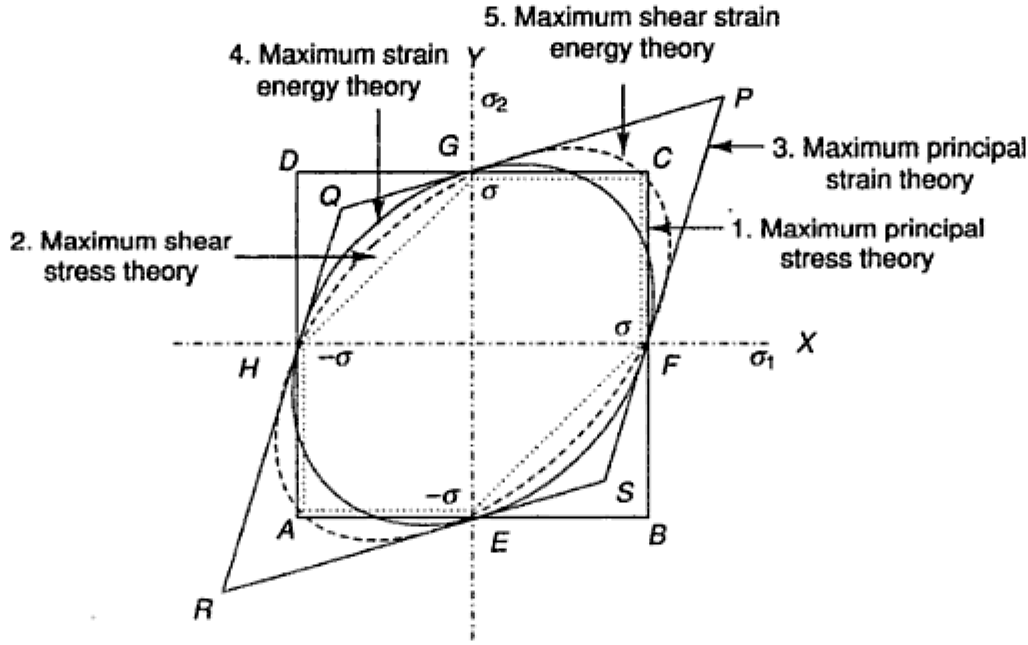


Fig. 3.10

Maximum principal stress theory According to maximum principal stress theory, the maximum principal stress σ_1 (or σ_2) must not exceed the elastic limit σ. Thus maximum value of σ_1 and σ_2 can be

$$\sigma_1 = \sigma,\ \sigma_2 = \sigma,\ \sigma_1 = -\sigma \text{ and } \sigma_2 = -\sigma$$

This provides a square boundary *ABCD* as shown in Fig. 3.10

Maximum shear stress theory For alike type of stresses when both are tensile or compressive i.e. both lie in first or third quadrant and the stress in the third perpendicular plane is assumed zero,

$$\sigma_1 = (\sigma - 0) = \sigma,$$

Similarly, $\sigma_2 = \sigma,\ \sigma_1 = -\sigma$ or $\sigma_2 = -\sigma$

These values generate the boundary lines *FC, CG, HA* and *AE* in the first and third quadrants.

When the principal stresses are of opposite type, i.e. if one is tensile, the other is compressive, then

$$\sigma_1 - \sigma_2 = \pm\,\sigma$$

If σ_2 is assumed negative, it can be written as

$$\sigma_1 - (-\sigma_2) = +\,\sigma \quad \text{or} \quad \sigma_1 + \sigma_2 = +\,\sigma$$

This provides a straight boundary *EF* in the fourth quadrant.

When σ_1 is assumed negative, the equation will be,

$$(-\sigma_1) - \sigma_2 = -\,\sigma \quad \text{or} \quad -\sigma_1 - \sigma_2 = -\,\sigma$$

This provides the boundary *GH* in the second quadrant.

Thus the boundary for this criterion is *AEFCGHA*

Maximum principal strain theory In case of two-dimensional principal strain theory, we have

$$\sigma_1 - v\sigma_2 = \pm\,\sigma$$

For like principal stresses, the limits are provided by

$$\sigma_1 - v\sigma_2 = \sigma,\ \sigma_2 - v\sigma_1 = \sigma,\ \sigma_1 - v\sigma_2 = -\sigma \text{ and } \sigma_2 - v\sigma_1 = -\sigma$$

For like principal stresses, the lines generated are *FP, PG, HR* and *RE.*

For unlike stresses, the lines are *GQ, QH, ES* and *SH* respectively in a similar way.

Maximum strain energy theory In maximum strain energy theory, the equation in two-dimensional system is

$$\sigma_1^2 + \sigma_2^2 - 2v\sigma_1\sigma_2 = \sigma$$

which is the equation of an ellipse with axes at 45° to the axes. It passes through the points *EFGH*.

Maximum shear strain energy theory In this, the equation in the two-dimensional system is

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma^2$$

which is again an equation of ellipse and is plotted in the figure.

Summary

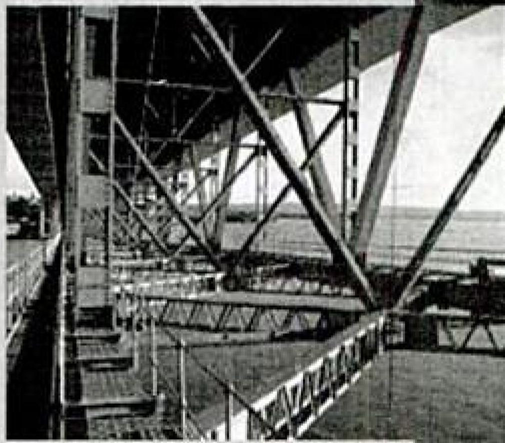
- The work done by the load in straining the material of a body is stored within it in the form of energy known as *strain energy*.
- Resilience is defined as the strain energy per unit volume in simple tension or compression and is equal to $\sigma^2/2E$.
- Proof resilience is the value of resilience at the elastic limit or at proof stress. It is also known as the *Modulus of Resilience*.
- Modulus of toughness is the strain energy per unit volume required to cause the material to rupture. It is the area under the stress-strain diagram upto rupture point.
- Strain energy, $U = \frac{1}{2}.P.\Delta = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume} = \frac{\sigma^2}{2E} \times \text{volume}$
- Strain energy per unit volume (3-D),
$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$
- Shear strain energy per unit volume $= \frac{\tau^2}{2G}$
- Shear strain energy per unit volume (3-D)
$$U = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$
- Maximum stress induced in a body due to suddenly applied load is twice the stress induced by the load of the same magnitude applied gradually.
- In case of impact or shock loading, $\sigma = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right)$
- Various theories of failure are:
  - (i) Maximum principal stress theory (Rankine's theory)
  - (ii) Maximum shear stress theory (Guest's and Tresca's theory)
  - (iii) Maximum principal strain theory (St. Venant's theory)
  - (iv) Maximum strain energy theory (Haigh's theory)
  - (v) Maximum shear strain energy theory (Mises' and Henkey's theory)

Review Questions

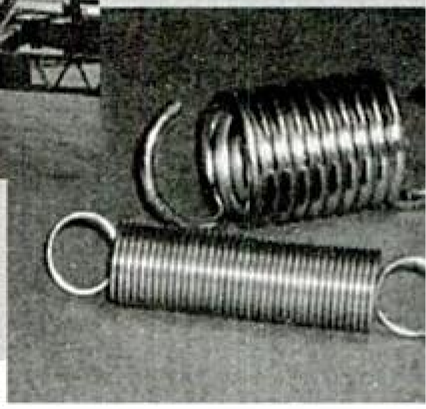
1. What is strain energy of a material? Derive the expressions for the same in different forms.
2. Define the terms: resilience, proof resilience, modulus of resilience.
3. Show that the proof resilience is the area under the stress– strain diagram upto elastic limit of a material.

4. Derive expressions for the strain energy in a three-dimensional stress system.
5. What is shear strain energy? Find its value per unit volume of the material.
6. Derive the relation for shear strain energy for a three-dimensional stress system.
7. What is the value of maximum stress induced in a body when the load is applied suddenly?
8. Deduce the relation for stress in case of impact and shock loading.
9. What are the main theories of failure for a material? Explain their relative use.
10. Give an account of graphical representation of various theories of failure.
11. A load of 22 kN is lowered by a steel rope at the rate of 750 mm/s. The diameter of the rope is 28 mm. When the length of the rope unwound is 12 m, the rope suddenly gets jammed. Find the instantaneous stress developed in the rope. Also calculate the instantaneous elongation of the rope. $E = 205$ GPa. (187.1 MPa, 10.95 mm)
12. A weight of 2 kN falls 24 mm on to a collar fixed to a steel bar that is 14 mm in diameter and 5.5 m long. Determine the maximum stress induced in the bar. $E_s = 205$ GPa. (166 MPa)
13. A weight of 800 N falls 30 mm on to a collar fixed to a steel bar of 1.2-m length. The steel bar is of 24-mm diameter for half of its length and 12 mm for the rest half. Determine the maximum stress and the extension in the bar. $E_s = 205$ GPa. (347.7 MPa; 1.272 mm)
14. A lift is operated by two 20-m long ropes and consisting of 30 wires of 1.5-mm diameter. The weight of the cage is 1 kN and the rope weighs 3.6 N/m length. Determine the maximum load that the lift can carry if it drops through 120 mm during operations. E (rope) = 78 GPa and allowable stress = 125 MPa. (1.188 kN)
15. A vertical tie rod consists of a 3-m long and of 24-mm diameter steel rod encased throughout in a brass tube of 24-mm internal diameter and 36-mm external diameter. The rod is rigidly fixed at the top end. The composite tie rod is suddenly loaded by a weight of 13.5 kN falling freely through 6 mm before being stopped by the tie. Determine the maximum stresses in steel and the brass. $E_s = 205$ GPa and $E_b = 98$ GPa. (143.8 MPa; 68.76 MPa)
16. An axial pull of 20 kN alongwith a shear force of 15 kN is applied to a circular bar of 20 mm diameter. The elastic limit of the bar material is 230 MPa and the Poisson's ratio, $\nu = 0.3$. Determine the factor of safety against failure based on
    (a) maximum shear stress theory
    (b) maximum strain energy theory
    (c) maximum principal strain energy theory
    (d) maximum shear strain energy theory (2; 2.3; 2.37; 2.2)

4



SHEAR FORCE AND BENDING MOMENT



4.1 INTRODUCTION

A structural element which is subjected to load transverse to its axis is known as a beam. In general, a beam is either free from any axial force or its effect is negligible. Analysis of beams involves the determination of shear force, bending moment and the deflections at various sections. This chapter deals with the finding of shear force and bending moment at a section of different kinds of beams. Analysis of beams to find the deflections is dealt in a later chapter.

Usually, a beam is considered horizontal and the load vertical. Other cases are considered as exceptions. A concentrated load is assumed to act at a point, though in practice it may be distributed over a small area. A distributed load is one which is spread over some length of the beam. The rate of loading may be uniform or may vary from point to point.

4.2 TYPES OF SUPPORTS AND BEAMS

A beam may have the following kinds of supports:

(i) Roller support When a beam rests on a sliding surface such as a roller or any flat surface like a masonry wall, the support is known as a roller support (Fig. 4.1*a*). A roller support can sustain a force only normal to its surface as the possible movement on the supporting surface does not allow any resistance in that direction. Thus it has reaction normal to the surface only and the reaction along the rolling surface is zero. A roller support allows the rotation of the body.

(ii) Hinged support In a hinged support no translational displacement of the beam is possible, however, it is free to rotate (Fig. 4.1*b*). A hinged support can sustain reactions in vertical as well as horizontal directions.

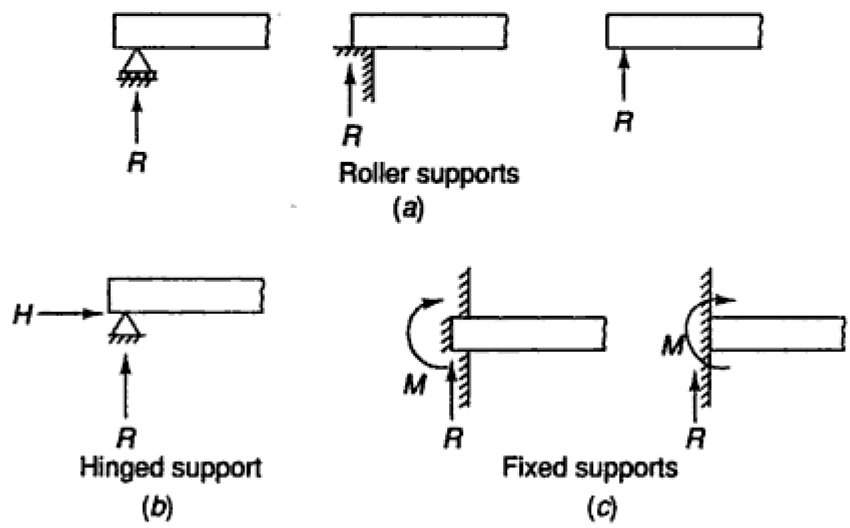


Fig. 4.1 Type of supports (as conventionally drawn)

(iii) Fixed or encastre or built-in support A beam built into a rigid support which does not allow any type of movement or rotation is known as *fixed* or *encastre* or *built-in support* (Fig. 4.1*c*). A fixed support exerts a fixing moment and a reaction on the beam.

Depending upon the type of support, the beams are classified as follows:

(i) *Simply supported beam* When both the supports of the beam are roller supports or one support is roller and the other hinged, the beam is known as a *simply supported beam* (Fig. 4.2*a*).

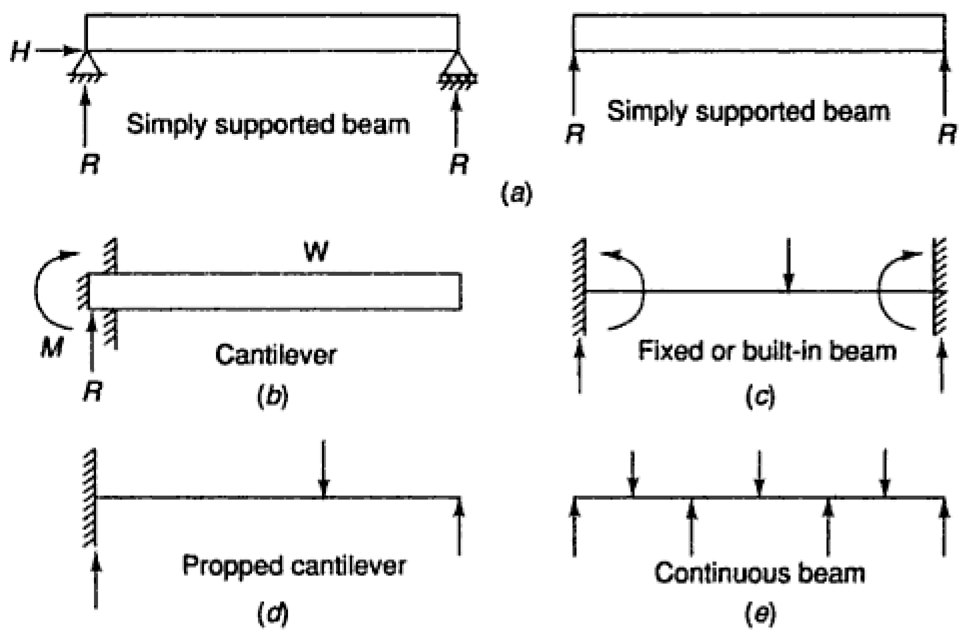


Fig. 4.2

(ii) A beam with one end fixed and the other end free is called a *cantilever* (Fig. 4.2*b*). There is a vertical reaction and moment at the fixed end (known as *fixing moment*).

(iii) A beam with both ends fixed is known as *fixed beam* (Fig. 4.2*c*).

(iv) Beams with one end fixed and the other simply supported are known as *propped cantilever* (Fig. 4.2*d*).

(v) Beams supported at more than two sections are known as *continuous beams* (Fig. 4.2*e*).

In beams with *hinge* joints, the bending moment at the hinge is taken to be zero.

Generally beams with more than two reaction components cannot be analysed using the equations of static equilibrium alone and are known as *statically indeterminate beams.*

4.3 SHEAR FORCE

Shear force is the unbalanced vertical force on one side (to the left or right) of a section of a beam and is the sum of all the normal forces on one side of the section. It also represents the tendency of either portion of the beam to slide or shear laterally relative to the other. Remember that a *force* at a section means a force of a certain magnitude acting at that point whereas the *shear force* at a section means the sum of all the forces on one side of the section.

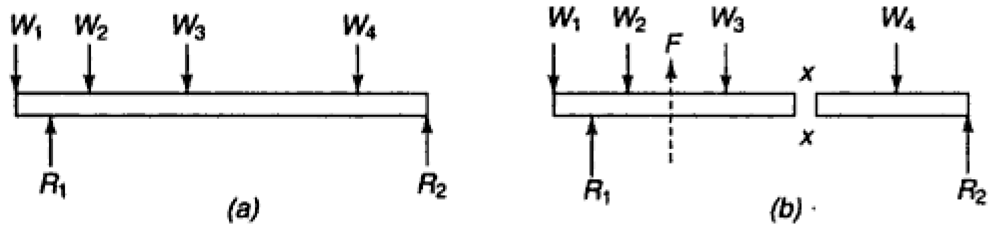


Fig. 4.3

Consider the beam as shown in Fig. 4.3*a*. It is simply supported at two points and carries four loads. The reactions at the supports are R_1 and R_2. Now if the beam is imagined to cut at section *x-x* into two portions (Fig. 4.3*b*), the resultant of all the forces (loads as well as reaction of support) to the left of the section is F (assuming upwards). Also, as the beam is in equilibrium, the resultant of the forces to the right of *x-x* must also be F downwards. The force F is known as shear or shearing force (S.F.)

Shear force is considered positive when the resultant of the forces to the left of a section is upwards or to the right downwards.

A *shear force diagram* (*SFD*) shows the variation of shear force along the length of a beam.

4.4 BENDING MOMENT

Bending moment at a section of a beam is defined as the algebraic sum of the moments about the section of all the forces on one side of the section.

If the moment M about the section *x-x* of all the forces to the left is clockwise (Fig. 4.4), then for the equilibrium, the moment of the forces to the right of *x-x* must be M counter-clockwise.

Bending moment is considered positive if the moment on the left portion is clockwise or on the right portion counter-clockwise. This is usually referred as *sagging* bending moment as it tends to cause concavity upwards. A bending moment causing

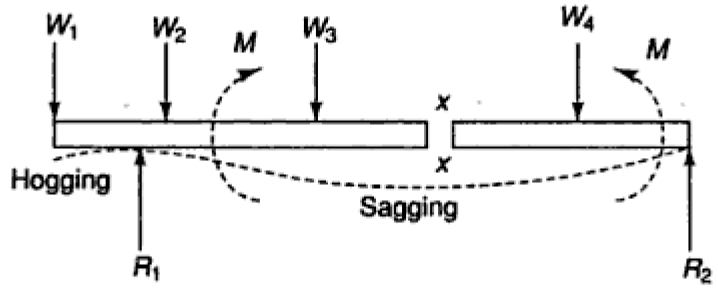


Fig. 4.4

convexity upwards is taken as negative bending moment and is called *hogging* bending moment.

A *bending moment diagram (BMD)* shows the variation of bending moment along the length of a beam.

4.5 RELATION BETWEEN *W*, *F* AND *M*

Consider a small length δx cut out from a loaded beam at a distance x from a fixed origin O (Fig. 4.5). Let

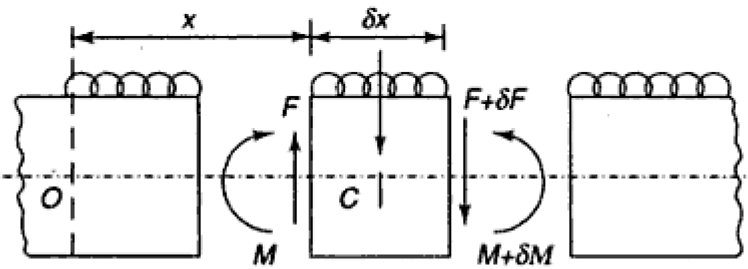


Fig. 4.5

w = mean rate of loading on the length δx
F = shear force at the section x
$F + \delta F$ = shear force at the section $x + \delta x$
M = bending moment at the section x
$M + \delta M$ = bending moment at the section $x + \delta x$

Total load on the length $\delta x = w.\delta x$ acting approximately through the centre C (if the load is uniformly distributed, it will be exactly acting through C).

For equilibrium of the element of length δx, equating vertical forces,

$$F = w\delta x + (F + \delta F) \quad \text{or} \quad w = -\frac{dF}{dx} \tag{4.1}$$

that is, rate of change of shear force (or slope of the shear force curve) is equal to intensity of loading.

Taking moments about C, $M + F.\dfrac{\delta x}{2} + (F + \delta F).\dfrac{\delta x}{2} - (M + \delta M) = 0$

Neglecting the product and squares of small quantities,

$$F = \frac{dM}{dx} \tag{4.2}$$

i.e. rate of change of bending moment is equal to the shear force.

The point of zero bending moment, i.e. where the type of bending changes from sagging to hogging is called a point of *inflection* or *contraflexure*.

Integrating Eq. 4.1 between two values of x,

$$F_a - F_b = \int_a^b w dx$$

which is the area under the load distribution diagram.

Similarly, integrating Eq. 4.2 between two values of x,

$$M_b - M_a = \int_a^b F dx$$

This shows that the variation of bending moment between two sections is equal to the area under the shear force diagram.

Also as $F = \frac{dM}{dx}$, $w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}$

4.6 SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR CANTILEVERS

A cantilever may carry concentrated or uniformly distributed loads.

Concentrated Loads

Assume a cantilever of length l carrying a concentrated load W at its free end as shown in Fig. 4.6*a*.

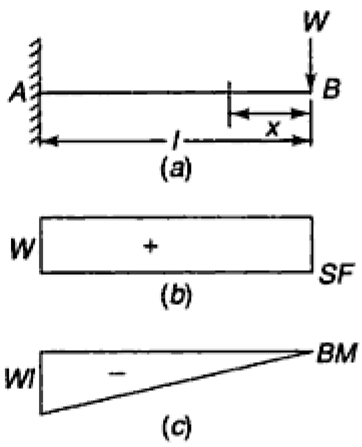


Fig. 4.6

Shear force diagram Consider a section at a distance x from the free end. The force to the right of the section is W downwards and is constant along the whole length of the beam or for all values of x. Therefore, the shear force will be considered positive and the shear force diagram is a horizontal straight line as shown in Fig. 4.6*b*.

Bending moment diagram Taking moments about the section, $M = W.x$

As the moment on the right portion of the section is clockwise, the bending moment diagram is negative. The bending moment can also be observed as hogging, and thus negative. The bending moment diagram is thus an inclined line increasing with the value of x (Fig. 4.6*c*).

Maximum bending moment = $W.l$ at the fixed end.

Reaction and the fixing moment From equilibrium conditions, the reaction at the fixed end is W and the fixing moment applied at the fixed end = Wl

Uniformly Distributed Load

Assume a cantilever of length l carrying a uniformly distributed load w per unit length across the whole span as shown in Fig. 4.7*a*.

Shear force diagram Consider a section at a distance x from the free end. The force to the right of the section is wx downwards and varies linearly along the whole length of the beam. Therefore, the shear force is positive and the shear force diagram is a straight line as shown in Fig. 4.7*b*.

Bending moment diagram The force wx to the right of section can be assumed to be acting as a concentrated load at a point at a distance $x/2$ from the free end.

Taking moments about the section, $M = wx.\frac{x}{2} = \frac{wx^2}{2}$

As the moment on the right portion is clockwise, the bending moment diagram is negative (hogging). The bending moment diagram is parabolic and increases with the value of x (Fig. 4.7*c*).

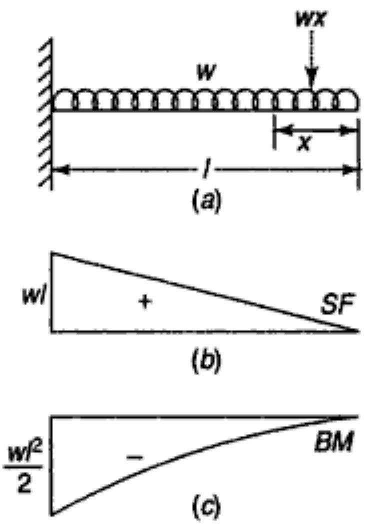


Fig. 4.7

Maximum bending moment = $\frac{wl^2}{2}$ at the fixed end.

Example 4.1 *A cantilever is loaded as shown in Fig. 4.8a. Draw the shear force and bending moment diagrams.*

Solution

Shear force diagram

- Portion *CD*

  Consider a section at a distance x from the free end.

  The force to the right of the section, $F_x = 6$ kN (constant)
- Portion *BC*: $F_x = 6 + 6 = 12$ kN (constant)
- Portion *AB*: $F_x = 6 + 6 + 4 = 16$ kN (constant)

Shear force diagram thus consists of several rectangles having different ordinates (Fig. 4.8*b*). It can be observed that the shear force undergoes a sudden change when passing through a load point.

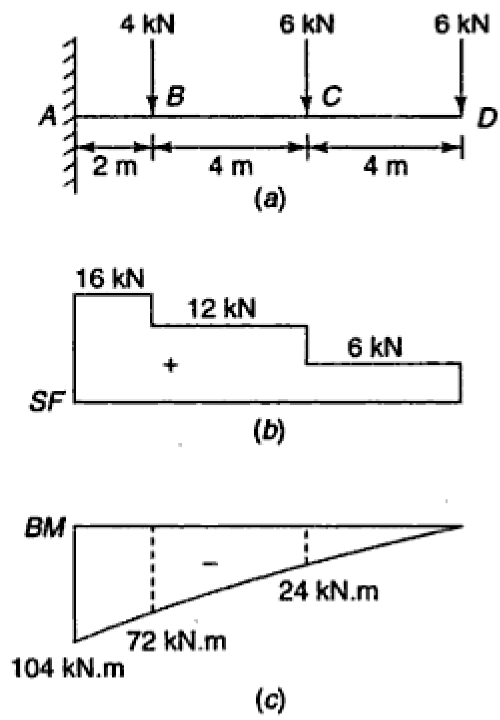


Fig. 4.8

Bending moment diagram

- Portion *CD*: Taking moments about a section, $M = 6x$, i.e. it is linear.

  At *D*, $x = 0$ and $M_d = 0$; At *C*, $x = 4$ m and $M_c = 24$ kN.m
- Portion *BC*: Taking moments about a section, $M_x = 6x + 6(x - 4)$ (linear)

  At *C*, $x = 4$ m and $M_c = 24$ kN.m; At *B*, $x = 8$ m and $M_b = 72$ kN.m
- Portion *AB*: $M_x = 6x + 6(x - 4) + 4(x - 8)$ (linear)

  At *B*, $x = 8$ m and $M_c = 72$ kN.m; At *A*, $x = 10$ m and $M_b = 104$ kN.m

The bending moment diagram is a series of straight lines between the loads (Fig. 4.8*c*).

Example 4.2 *Figures 4.9 a, b and c show three cantilevers loaded in different ways. Draw the shear force and bending moment diagrams in each case.*

Solution

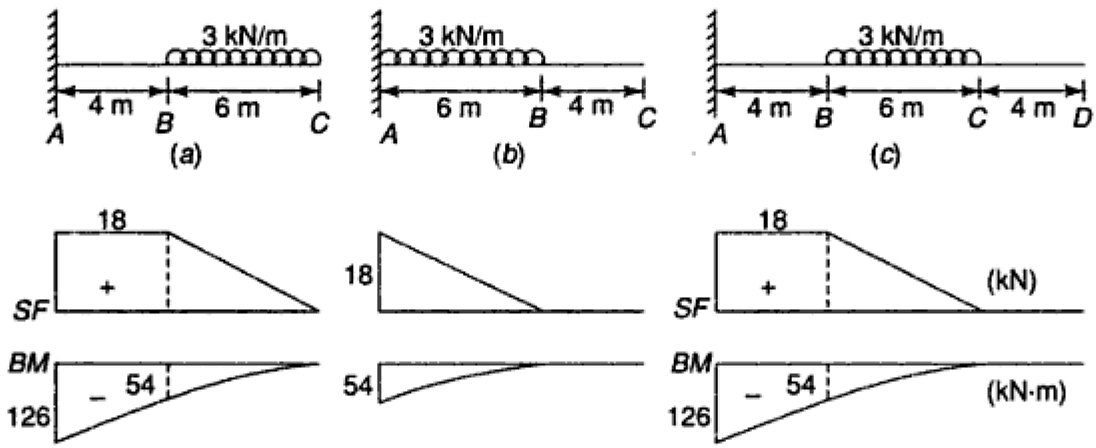


Fig. 4.9

(i) *Shear force diagram*

- Portion *BC*: $F_x = 3x$ (linear); $F_c = 0$; $F_b = 18$ kN;
- Portion *AB*: $F_x = 18$ kN (constant)

Bending moment diagram

- Portion *BC*: $M_x = 3x.(x/2) = 1.5x^2$ (parabolic); $M_c = 0$; $M_b = 54$ kN.m
- Portion *AB*: $M_x = 18(x - 3)$ (linear); $M_b = 54$ kN.m; $M_a = 126$ kN.m

Shear force and bending moment diagrams are shown below the load diagram in the Fig. 4.9*a*.

(ii) *Shear force diagram*

- Portion *BC*: $F_x = 0$; $F_c = F_b = 0$
- Portion *AB*: At distance x from *B*, $F_x = 3x$ (linear); $F_b = 0$; $F_a = 18$ kN.m

Bending moment diagram

- Portion *BC*: $M_x = 0$; $M_b = 0$
- Portion *AB*: At distance x from *B*, $M_x = 3x.(x/2) = 1.5x^2$ (parabolic)

$$M_b = 0;\ M_a = 54 \text{ kN.m}$$

Shear force and bending moment diagrams are shown below the load diagram in the Fig. 4.9*b*.

(iii) There will not be any shear force and bending moment in the portion *CD* and for the portion *AC*, the shear force and bending moment diagrams will as in case (i).

Example 4.3 *A cantilever is loaded as shown in Fig. 4.10a. Draw the shear force and bending moment diagrams.*

Solution *Shear force diagram*

- Portion *EG*: $F_x = 4$ kN (constant);

- Portion *EG*: $F_x = 4$ kN (constant);
- Portion *DE*: $F_x = 4 + 2(x - 4)$ (linear); $F_e = 4$ kN; $F_d = 10$ kN
- Portion *CD*: $F_x = 4 + 2(x - 4) + 6$; (linear); $F_d = 16$ kN; $F_c = 22$ kN
- Portion *BC*: $F_x = 22 + 4$; (constant); $F_c = F_b = 26$ kN;
- Portion *AB*: $F_x = 26 + 6$; (constant); $F_a = F_b = 32$ kN;

Shear force diagram has been shown in Fig. 4.10*b*.

Bending moment diagram

- Portion *EG*: $M_x = 4x$ (linear); $M_g = 0$; $M_e = 16$ kN.m
- Portion *DE*: $M_x = 4x + \frac{2(x-4)^2}{2}$ ...(parabolic); $M_e = 16$ kN.m ; $M_d = 37$ kN.m

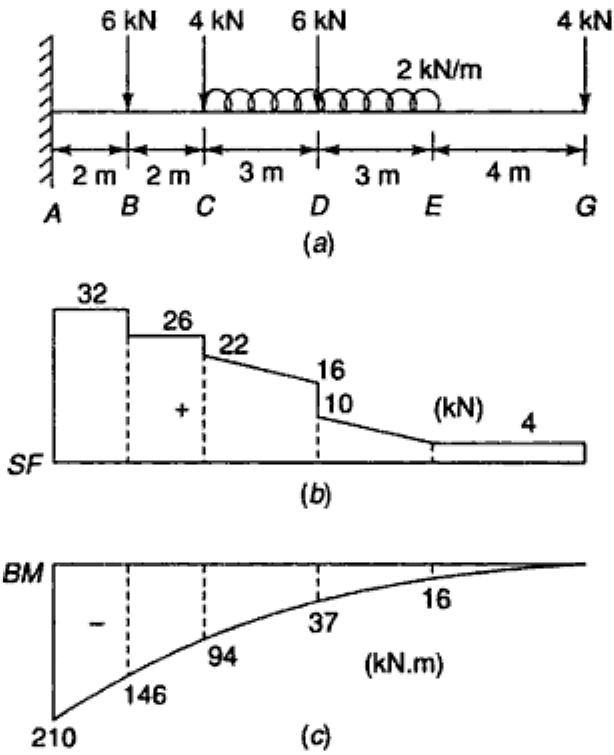


Fig. 4.10

- Portion *CD*: $M_x = 4x + \frac{2(x-4)^2}{2} + 6(x-7)$ (parabolic)

  $M_{d(x=7)} = 37$ kN.m; $M_{c(x=10)} = 94$ kN.m
- Portion *BC*: $M_x = 4x + 2 \times 6(x-7) + 6(x-7) + 4(x-10)$ (linear)

  $M_{c(x=10)} = 94$ kN.m; $M_{b(x=12)} = 146$ kN;
- Portion *AB*: $M_x = 4x + 2 \times 6(x-7) + 6(x-7) + 4(x-10) + 6(x-12)$ (linear)

  $M_{b(x=12)} = 146$ kN.m; $M_{a(x=14)} = 210$ kN

Bending moment diagram has been shown in Fig. 4.10*c*.

Example 4.4 *A cantilever is loaded with distributed load of varying intensity with zero load at the free end as shown in Fig. 4.11a. Draw the shear force and bending moment diagrams.*

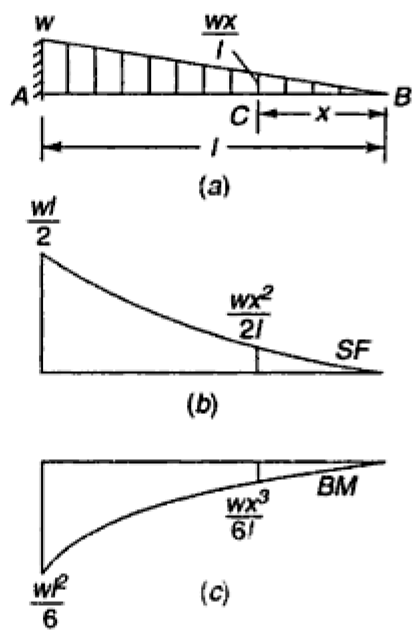


Fig. 4.11

Solution

Intensity of loading at any cross-section *C* at a distance *x* from free end = $\frac{w}{l}.x$

Shear force diagram

At a distance *x* from *B*, $F_x = \frac{1}{2}\frac{wx}{l}.x = \frac{wx^2}{2l}$ (parabolic); $F_b = 0$; $F_a = \frac{wl}{2}$

Shear force diagram is shown in Fig. 4.11*b*.

Bending moment diagram

Bending moment at C = load on $CB \times$ distance of centre of load

= (Average intensity × distance CB) × $CB/3$

$$= \frac{wx}{2l}.x.\frac{x}{3} = \frac{wx^3}{6l} \qquad \text{(cubic)}$$

or by differentiating the expression for shear force, i.e. $\frac{d}{dx}\left(\frac{wx^2}{2l}\right) = \frac{wx^3}{6l}$

$$M_b = 0; M_a = \frac{wl^2}{6}$$

(The expression for bending moment can also be found by integrating the expression for shear force, i.e., $M_x = \frac{wx^2}{6l} + C$ and at B; $x = 0$; $C = M_b = 0$)

Bending moment diagram is shown in Fig. 4.11*c*.

Note that the total distributed load acting on any portion is equal to the area of the load diagram on that portion.

Example 4.5 *A cantilever has distributed load of varying intensity with zero at the fixed end and* w *at the free end as shown in Fig. 4.12a. Draw the shear force and bending moment diagrams.*

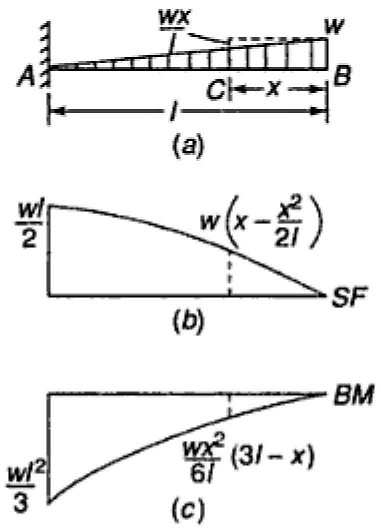


Fig. 4.12

Solution Intensity of loading at any cross-section C at a distance x from free end

$$= \frac{w}{l}(l - x) = w - \frac{wx}{l}$$

Shear force diagram

At a distance x from B,

F_x = Area of rectangle on CB – area of upper small triangle on CB

$$= wx - \frac{wx}{l}.\frac{x}{2} = w\left(x - \frac{x^2}{2l}\right) \text{ (parabolic)}$$

(or by integrating the expression for load)

$F_b = 0$; $F_a = wl/2$

Shear force diagram is shown in Fig. 4.12*b*.

Bending moment diagram

Bending moment at $C = wx.\frac{x}{2} - \frac{wx}{l}.\frac{x}{2}.\frac{x}{3} = \frac{wx^2}{6l}(3l - x)$ (Cubic)

(or by integrating the expression for shear force)

$$M_b = 0; M_a = \frac{wl^2}{3}$$

Bending moment diagram is shown in Fig. 4.12*c*.

Example 4.6 *A cantilever is loaded as shown in Fig. 4.13a. Draw the shear force and bending moment diagrams.*

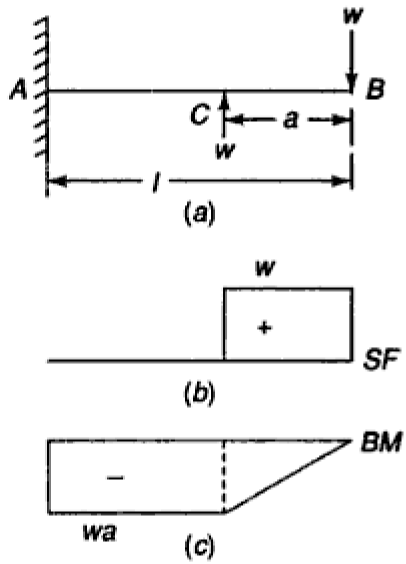


Fig.4.13

Solution *Shear force diagram*

- Portion *CB*: $F_x = W$ (constant)
- Portion *AC*: $F_x = W - W = 0$

Shear force diagram is shown in Fig. 4.13*b*.

Bending moment diagram

- Portion *CB*: $M_x = Wx$ (linear); $M_b = 0$; $M_c = W.a$
- Portion *AC*: $M_x = Wx - W(x - a) = Wa$ (constant)

Bending moment diagram is shown in Fig. 4.13*c*.

4.7 SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR SIMPLY SUPPORTED BEAMS

A simply supported beam may carry concentrated or uniformly distributed loads.

Concentrated Loads

Assume a simply supported beam of length l carrying a concentrated load W at a distance a from end A as shown in Fig. 4.14*a*.

Let the distance *CB* be b.

First it is required to find the reactions at the supports.

Taking moments about end $A = R_b.l - W.a = 0$

or $R_b = \dfrac{Wa}{l}$; Similarly, $R_a = \dfrac{Wb}{l}$

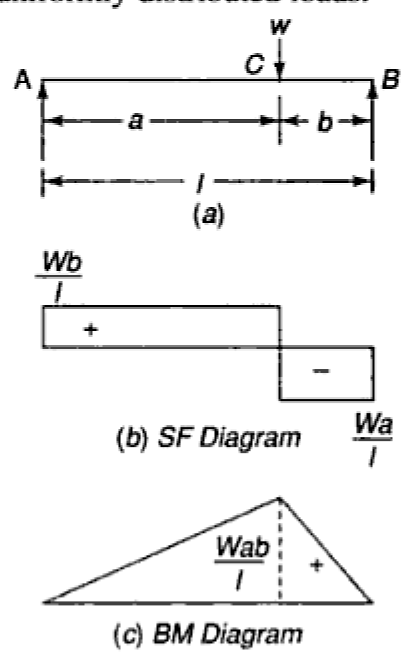


Fig. 4.14

Shear force diagram

- *Portion BC:* Consider a section at a distance x from the end B. The force to the right of the section is R_b upwards and is constant along the length upto point C on the beam. Therefore, the shear force will be negative and the shear force diagram is a horizontal straight line as shown in Fig. 4.14*b*.
- *Portion AC*: At a section at a distance x from the end B, the force to the right of section is

$$= R_b - W = \frac{Wa}{l} - W = \frac{Wa - Wl}{l}$$

$$= -W\left(\frac{l-a}{l}\right) = -\frac{Wb}{l} = R_a \text{ (downwards)}$$

Thus, the shear force will be positive and the shear force diagram is a horizontal straight line as shown in Fig. 4.14*b*.

Note that the portion *AB* can also be taken first and the forces to the left of any section may be considered. In that case, the force to the left is R_a and upwards and the shear force positive, i.e. the same as before.

Bending moment diagram

Portion BC: Consider a section at a distance x from the end *B*.

Taking moments about the section, $M = R_b.x = \dfrac{Wax}{l}$

$$M_b = 0; M_c = Wab/l$$

As the moment on the right portion of the section is counter-clockwise, the bending moment is positive. The bending moment can also be observed as sagging, and thus positive. Therefore the bending moment is linear and increases with the value of x (Fig. 4.14*c*).

Portion AC: Consider a section at a distance x from the end *A*.

$$M = R_a.x = \frac{Wbx}{l};\ M_a = 0;\ M_c = Wab/l$$

The moment on the left portion of the section is clockwise, the bending moment is positive. The bending moment can also be observed as sagging, and thus positive.

- If the load is at the midspan, $a = b = l/2$

  The bending moment at the midpoint, $M = \dfrac{W(l/2)(l/2)}{l} = \dfrac{Wl}{4}$ which is maximum for any position of the load on beam.

Uniformly Distributed Load

Assume a simply supported beam of length l carrying a uniformly distributed load w per unit length as shown in Fig. 4.15*a*.

Total load = wl; $R_a = R_b = wl/2$

Shear Force Diagram

At a section at a distance x from *A*

$$F_x = R_a - wx = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right) \text{ (linear);}$$

$$F_{a(x=0)} = \frac{wl}{2};\ F_{b(x=l)} = -\frac{wl}{2}$$

Shear force diagram is shown in Fig. 4.15*b*.

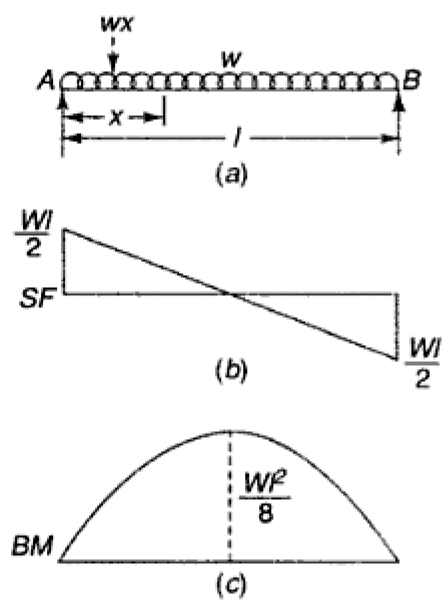


Fig. 4.15

Bending moment diagram

The bending moment at a section is found by treating the distributed load as acting at its centre of gravity.

$$M_x = R_a x - wx.\frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(l-x) \qquad \text{(parabolic)}$$

$$M_{a(x=0)} = 0;\ M_{b(x=l)} = 0;$$

For maximum value, $F = \frac{dM}{dx} = 0$ or $\frac{wl}{2} - wx = 0$ or $x = l/2$

Thus maximum bending moment, $M_{(x=l/2)} = \frac{wl^2}{8}$

Bending moment diagram is shown in Fig. 4.15*c*.

Uniformly Distributed Load with Equal Overhangs

Let w be the uniformly distributed load on the beam as shown in Fig. 4.16*a*.

As the overhangs are equal, $R_a = R_b = \frac{w(l+2a)}{2}$

Shear force diagram

- Portion *DA*: $F_x = -wx$ (linear); $F_d = 0$; $F_a = -wa$
- Portion *AB*: $F_x = -wx + \frac{w(l+2a)}{2}$ (linear); $F_{a(x=a)} = \frac{wl}{2}$; $F_{b(x=l+a)} = -\frac{wl}{2}$
- Portion *BE*: $F_x = -wx + \frac{w(l+2a)}{2} + \frac{w(l+2a)}{2} = -wx + w(l+2a)$ (linear)

$$F_{b(x=l+a)} = wa;\ F_{e(x=l+2a)} = 0;$$

Shear force diagram is shown in Fig. 4.16*b*.

Bending moment diagram

- Portion *DA*: $M_x = -\frac{wx^2}{2}$ (parabolic) ; $M_d = 0$; $M_a = -\frac{wa^2}{2}$
- Portion *AB*: $M_x = -\frac{wx^2}{2} + \frac{w(l+2a)}{2}(x-a)$ (parabolic)

$$M_a = -\frac{wa^2}{2};\ M_{b(x=l+a)} = -\frac{wa^2}{2}$$

- Portion *BE*: Bending moment will be reducing to zero in a parabolic manner at *E*. It is convenient to consider it from end *E*. Then $M_x = -wx^2/2$.

At midpoint *C*,

$$M_{c(x=a+l/2)} = -\frac{w(a+l/2)^2}{2} + \frac{w(l+2a)}{2}.\frac{l}{2}$$

$$= -\frac{w}{2}\left(a^2 + \frac{l^2}{4} + al - \frac{l^2}{2} - al\right) = \frac{w}{8}(l^2 - 4a^2)$$

The shape of bending moment curve between A and B will depend upon the value of $(l^2 - 4a^2)$.

(i) If $(l^2 - 4a^2) < 0$, i.e. when $l < 2a$, M_c is negative which means bending moment will be negative throughout (Fig. 4.16c).

(ii) If $(l^2 - 4a^2) = 0$ i.e. when $l = 2a$, M_c is zero which means bending moment will just touch the span at the midpoint (Fig. 4.16*d*).

(iii) If $(l^2 - 4a^2) > 0$, i.e. when $l > 2a$, M_c is positive which means there are to be two points of contraflexture between A and B which can be found by putting the expression for bending moment between A and B equal to zero (Fig. 4.16*e*), i.e.

$$-\frac{wx^2}{2} + \frac{w(l+2a)}{2}(x-a) = 0$$

or $x^2 - (2a + l)x = -a(2a + l)$

Adding $(a + l/2)^2$ on both sides,

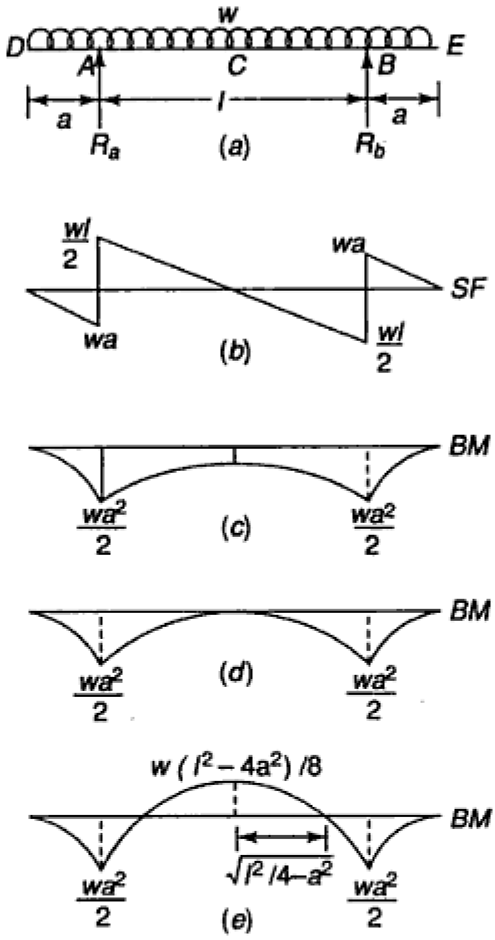


Fig. 4.16

$$x^2 - 2\left(a + \frac{l}{2}\right)x + \left(a + \frac{l}{2}\right)^2 = -a(2a + l) + \left(a + \frac{l}{2}\right)^2$$

or $$\left[x - \left(a + \frac{l}{2}\right)\right]^2 = \frac{l^2}{4} - a^2$$

or $$x - \left(a + \frac{l}{2}\right) = \pm\sqrt{\frac{l^2}{4} - a^2}$$

or $$x = \left(a + \frac{l}{2}\right) \pm \sqrt{\frac{l^2}{4} - a^2}$$

As $a + l/2$ is the distance DC, the points of contraflexture are at distance $\pm\sqrt{\frac{l^2}{4} - a^2}$ from the midpoint of the beam.

Couples

Let a beam be subjected to a couple M as shown in Fig. 4.17*a*.

To find reactions at the supports, take moments about B,

$R_a . l = M$ or $R_a = M/l$ and $R_b = -R_a = -M/l$

The shear force diagram is a rectangle as shown in Fig. 4.17*b*.

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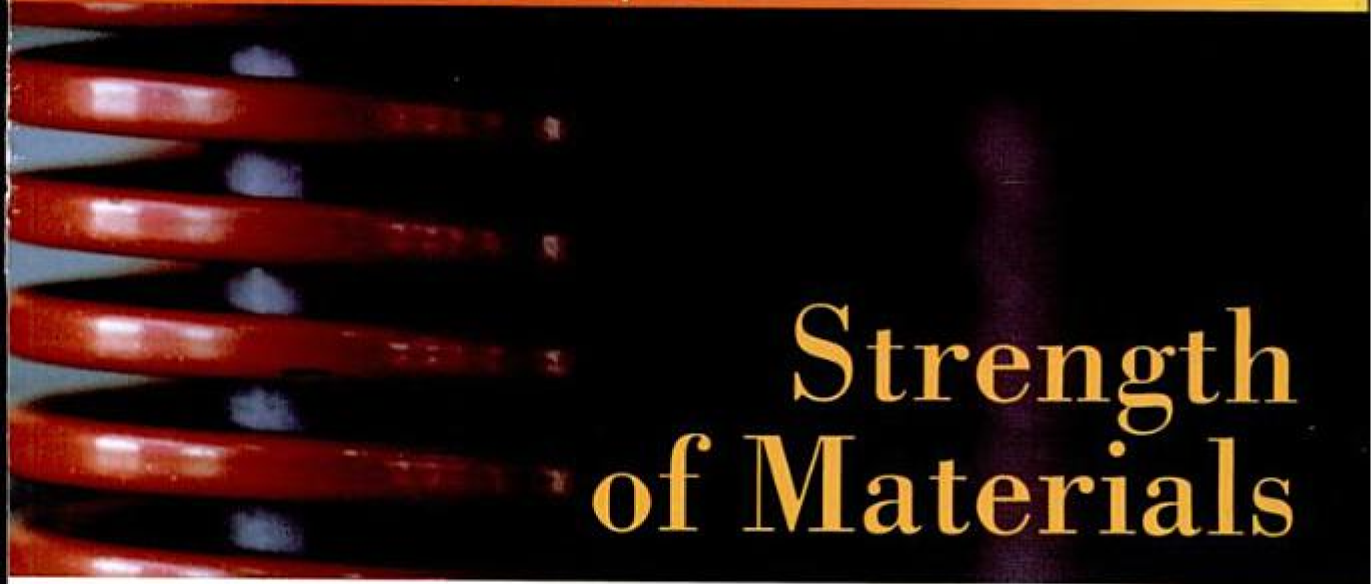

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